

# UPPER BOUND ON SURFACE MEASURE OF NODAL SETS

GCXDOMSECT

## 1. EIGENFUNCTIONS IN THE COMPLEX DOMAIN

In this section we consider eigenfunctions of real analytic Riemannian manifolds. On a real analytic Riemannian manifold  $(M, g)$  of dimension  $m$ , we analytically continue an orthonormal basis  $\{\varphi_{\lambda_j}\}$  of eigenfunctions,

$$\Delta_g \varphi_{\lambda_j} = \lambda_j^2 \varphi_{\lambda_j}, \quad \langle \varphi_{\lambda_j}, \varphi_{\lambda_k} \rangle = \delta_{jk}, \quad (\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \dots),$$

into the complexification  $M_{\mathbb{C}}$  of  $M$ . As recalled in §2, eigenfunctions admit analytic continuations  $\varphi_{\lambda_j}^{\mathbb{C}}$  to a maximal uniform 'Grauert tube'

MTAU

$$(1.1) \quad M_{\tau} = \{\zeta \in M_{\mathbb{C}}, \sqrt{\rho}(\zeta) < \tau\}$$

independent of  $\lambda_j$ , where the radius is measured by the Grauert tube function  $\sqrt{\rho}(\zeta)$  corresponding to  $g$  (see §2: [LS1, GS1]). As discussed below, given the metric  $g$  there is a relatively canonical identification of  $M_{\varepsilon}$  with a ball bundle  $B_{\varepsilon}^* \subset T^*M$ , so that one may view  $M_{\varepsilon}$  as phase space with a complex structure. The modulus squares

HUSIMI

$$(1.2) \quad |\varphi_j^{\mathbb{C}}(\zeta)|^2 : M_{\varepsilon} \rightarrow \mathbb{R}_+$$

are sometimes known as Husimi functions. They are holomorphic extensions of  $L^2$ -normalized functions but are not themselves  $L^2$  normalized on  $M_{\varepsilon}$ . However, as will be discussed below, their  $L^2$  norms may on the Grauert tubes (and their boundaries) can be determined. One can then ask how the mass of the normalized Husimi function is distributed in phase space, or how the  $L^p$  norms behave.

The first motivation to analytically continue eigenfunctions is that it enables us to give a relatively simple proof of the Donnelly-Fefferman theorem on nodal hypersurface volumes. Let  $Z_{\varphi_{\lambda}}$  be the nodal set of an eigenfunction  $\varphi_{\lambda}$  of eigenvalue  $\lambda^2$ .

NODALBOUND

**Theorem 1.1.** *Let  $(M, g)$  be a real analytic Riemannian manifold. Then, there exists constants  $C, c > 0$  depending only on  $(M, g)$  so that*

$$c\lambda \leq \mathcal{H}^{n-1}(Z_{\varphi_{\lambda}}) \leq C\lambda.$$

The upper bound is based on proofs of nodal upper bounds in [Ze12, Ze15]. The key tool is the analytic continuation of the Poisson-wave kernel to Grauert tubes and its description as a Fourier integral operator with complex phase. The Hausdorff measure of the complex nodal set

CXN

$$(1.3) \quad Z_{\varphi_{\lambda}}^{\mathbb{C}} = \{\zeta \in (\partial\Omega)_{\mathbb{C}} : \psi_{\lambda_j}^{\mathbb{C}}(\zeta) = 0\}$$

gives an upper bound for the Hausdorff measure of the real nodal set. In the complex domain one may use the Poincaré-Lelong formula and a global Jensen type argument to give the upper bound (Section 9). The proof of the lower bound is also based on analytic continuation of eigenfunctions and is a new proof due to A. Brudnyi (Section 10).

Analytic continuation to the complex domain gives strong compactness properties to the sequence  $u_j = \frac{1}{\lambda_j} \log |\varphi_j^{\mathbb{C}}|^2$  of pluri-subharmonic functions. This gives rise to a new weak\* limit problem for eigenfunctions, discussed in Section 2. It is possible to solve the problem in ergodic cases (Section 7.7) and in integrable cases ([Ze1]). This allows one to determine the equidistribution of complex nodal sets in these settings, something which seems out of reach in the real domain.

We now give background on Grauert tubes, Szegő and Poisson kernels, on the analytic continuation of eigenfunctions and the wave group following [Ze12, Ze14, Ze15].

GRAUERT

## 2. GRAUERT TUBES AND COMPLEX GEODESIC FLOW

By a theorem of Bruhat-Whitney, a real analytic Riemannian manifold  $M$  admits a complexification  $M_{\mathbb{C}}$ , i.e. a complex manifold into which  $M$  embeds as a totally real submanifold. Corresponding to a real analytic metric  $g$  is a unique plurisubharmonic exhaustion function  $\sqrt{\rho}$  on  $M_{\mathbb{C}}$  (known as the Grauert tube function) satisfying two conditions (i) It satisfies the Monge-Ampère equation  $(i\partial\bar{\partial}\sqrt{\rho})^n = \delta_{M,g}$  where  $\delta_{M,g}$  is the delta function on  $M$  with density  $dV_g$  equal to the volume density of  $g$ ; (ii) the Kähler metric  $\omega_g = i\partial\bar{\partial}\rho$  on  $M_{\mathbb{C}}$  agrees with  $g$  along  $M$ . In fact,

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$$(2.1) \quad \sqrt{\rho}(\zeta) = \frac{1}{2i} \sqrt{r_{\mathbb{C}}^2(\zeta, \bar{\zeta})},$$

where  $r^2(x, y)$  is the square of the distance function and  $r_{\mathbb{C}}^2$  is its holomorphic extension to a small neighborhood of the anti-diagonal  $(\zeta, \bar{\zeta})$  in  $M_{\mathbb{C}} \times M_{\mathbb{C}}$ . In the case of flat  $\mathbb{R}^n$ ,  $\sqrt{\rho}(x + i\xi) = 2|\xi|$  and in general  $\sqrt{\rho}(\zeta)$  measures how far  $\zeta$  reaches into the complexification of  $M$ . The open Grauert tube of radius  $\tau$  is defined by  $M_{\tau} = \{\zeta \in M_{\mathbb{C}}, \sqrt{\rho}(\zeta) < \tau\}$ . We use the imprecise notation  $M_{\mathbb{C}}$  to denote the open complexification when it is not important to specify the radius.

The  $(1, 1)$  form  $\omega = \omega_{\rho} := i\partial\bar{\partial}\rho$  defines a Kähler metric on  $M_{\mathbb{C}}$ . The Grauert tubes  $M_{\tau}$  are strictly pseudo-convex domains in  $M_{\mathbb{C}}$ , whose boundaries  $\partial M_{\tau}$  are strictly pseudo-convex CR manifolds. The boundary is endowed with the contact form

alpha

$$(2.2) \quad \alpha = \frac{1}{i} \partial\rho|_{\partial M_{\tau}} = d^c \sqrt{\rho}.$$

**1. Analytic continuation of the exponential map.** The geodesic flow is a Hamiltonian flow on  $T^*M$ . In fact, there are two standard choices of the Hamiltonian. In PDE it is most common to define the (real) homogeneous geodesic flow  $g^t$  of  $(M, g)$  as the Hamiltonian flow on  $T^*M$  generated by the Hamiltonian  $|\xi|_g$  with respect to the standard Hamiltonian form  $\omega$ . This Hamiltonian is real analytic on  $T^*M \setminus 0$ . In Riemannian geometry it is standard to let the time of travel equal  $|\xi|_g$ ; this corresponds to the Hamiltonian flow of  $|\xi|_g^2$ , which is real analytic on all of  $T^*M$ . We denote its Hamiltonian flow by  $G^t$ . In general, we denote by  $\Xi_H$  the Hamiltonian vector field of a Hamiltonian  $H$  and its flow by  $\exp t\Xi_H$ . Both of the Hamiltonian flows

- $g^t = \exp t\Xi_{|\xi|_g}$ ;
- $G^t = \exp t\Xi_{|\xi|_g^2}$

are important in analytic continuation of the wave kernel. The exponential map is the map  $\exp_x : T^*M \rightarrow M$  defined by  $\exp_x \xi = \pi G^t(x, \xi)$  where  $\pi$  is the standard projection.

We denote by  $\text{inj}(x)$  the injectivity radius of  $(M, g)$  at  $x$ , i.e. the radius  $r$  of the largest ball on which  $\exp_x : B_r M \rightarrow M$  is a diffeomorphism to its image. Since  $(M, g)$  is real analytic,  $\exp_x t\xi$  admits an analytic continuation in  $t$  and the imaginary time exponential map

**EXP** (2.3) 
$$E : B_\varepsilon^* M \rightarrow M_{\mathbb{C}}, \quad E(x, \xi) = \exp_x i\xi$$

is, for small enough  $\varepsilon$ , a diffeomorphism from the ball bundle  $B_\varepsilon^* M$  of radius  $\varepsilon$  in  $T^* M$  to the Grauert tube  $M_\varepsilon$  in  $M_{\mathbb{C}}$ . We have  $E^* \omega = \omega_{T^* M}$ , where  $\omega = i\partial\bar{\partial}\rho$  and where  $\omega_{T^* M}$  is the canonical symplectic form; and also  $E^* \sqrt{\rho} = |\xi|$  [GSI, LS1]. It follows that  $E^*$  conjugates the geodesic flow on  $B^* M$  to the Hamiltonian flow  $\exp t\Xi_{\sqrt{\rho}}$  of  $\sqrt{\rho}$  with respect to  $\omega$ , i.e.

$$E(g^t(x, \xi)) = \exp t\Xi_{\sqrt{\rho}}(\exp_x i\xi).$$

**MAXG** 2. **Maximal Grauert tubes.** A natural definition of *maximal Grauert tube* is the maximum value of  $\varepsilon$  so that (2.3) is a diffeomorphism. We refer to this radius as the *maximal geometric tube radius*. But for purposes of this paper, another definition of maximality is relevant: the maximal tube on which all eigenfunctions extend holomorphically. A closely related definition is the maximal tube to which the Poisson kernel (5.11) extends holomorphically. We refer to the radius as the *maximal analytic tube radius*.

A natural question is to relate these notions of maximal Grauert tube has not been explored. We therefore define the radii more precisely:

- MAXGRAU** **Definition 2.1.** (1) The maximal geometric tube radius  $\tau_g$  is the largest radius  $\varepsilon$  for which  $E$  (2.3) is a diffeomorphism.
- (2) The maximal analytic tube radius  $\tau_{an}$   $M_{\tau_{an}} \subset M_{\mathbb{C}}$  is the maximal tube to which all eigenfunctions extend holomorphically and to which the anti-diagonal  $U(2i\tau, \zeta, \bar{\zeta})$  of the Poisson kernel admits an analytic continuation.

It is possible to prove that  $\tau_g = \tau_{an}$ . In §?? we sketch the proof that  $\tau_{an}$  is the maximal radius for which the coefficients of  $\Delta_g$  have holomorphic extensions. This radius is similar to the geometric radius, since the leading coefficients are geometric. But the coefficients of the first degree terms are not quite geometric in the same sense and at this time of writing the geometric radius has not been related to the maximal domain in which  $\Delta_g$  extends holomorphically. The proof which is based on holomorphic extensions of solutions of analytic PDE across non-characteristic hypersurfaces. We found a similar argument in [KS] in the case of locally symmetric spaces but employing additional arguments.

**EXAMPLES** 3. **Model examples.** .

We consider some standard examples to clarify these analytic continuations.

(i) **Complex tori:**

The complexification of the torus  $M = \mathbb{R}^m / \mathbb{Z}^m$  is  $M_{\mathbb{C}} = \mathbb{C}^m / \mathbb{Z}^m$ . The adapted complex structure to the flat metric on  $M$  is the standard (unique) complex structure on  $\mathbb{C}^m$ . The complexified exponential map is  $\exp_x^{\mathbb{C}}(i\xi) = z := x + i\xi$ , while the distance function  $r(x, y) = |x - y|$  extends to  $r_{\mathbb{C}}(z, w) = \sqrt{(z - w)^2}$ . Then  $\sqrt{\rho}(z, \bar{z}) = \sqrt{(z - \bar{z})^2} = \pm 2i|\text{Im } z| = \pm 2i|\xi|$ .

The complexified cotangent bundle is  $T^* M_{\mathbb{C}} = \mathbb{C}^m / \mathbb{Z}^m \times \mathbb{C}^m$ , and the holomorphic geodesic flow is the entire holomorphic map

$$G^t(\zeta, p_\zeta) = (\zeta + tp_\zeta, p_\zeta).$$

(ii)  $\mathbb{S}^n$  <sup>[GS1]</sup> The unit sphere  $x_1^2 + \cdots + x_{n+1}^2 = 1$  in  $\mathbb{R}^{n+1}$  is complexified as the complex quadric

$$\mathbb{S}_{\mathbb{C}}^n = \{(z_1, \dots, z_n) \in \mathbb{C}^{n+1} : z_1^2 + \cdots + z_{n+1}^2 = 1\}.$$

If we write  $z_j = x_j + i\xi_j$ , the equations become  $|x|^2 - |\xi|^2 = 1, \langle x, \xi \rangle = 0$ . The geodesic flow  $G^t(x, \xi) = (\cos t|\xi|x + (\sin t|\xi|)\frac{\xi}{|\xi|}, -|\xi|(\sin t|\xi|x + (\cos t|\xi|)\xi)$  on  $T^*\mathbb{S}^n$  complexifies to

$$\begin{aligned} G^t(Z, W) &= (\cos t\sqrt{W \cdot \overline{W}})Z + (\sin t\sqrt{W \cdot \overline{W}})\frac{W}{\sqrt{W \cdot \overline{W}}}, \\ &\quad -\sqrt{W \cdot \overline{W}}(\sin t\sqrt{W \cdot \overline{W}})Z + (\cos t\sqrt{W \cdot \overline{W}})W, \quad ((Z, W) \in T^*\mathbb{S}_{\mathbb{C}}^n). \end{aligned}$$

Here, the real cotangent bundle is the subset of  $T^*\mathbb{R}^{n+1}$  of  $(x, \xi)$  such that  $x \in \mathbb{S}^n, x \cdot \xi = 0$  and the complexified cotangent bundle  $T^*\mathbb{S}_{\mathbb{C}}^n \subset T^*\mathbb{C}^{n+1}$  is the set of vectors  $(Z, W) : Z \cdot W = 0$ . We note that although  $\sqrt{W \cdot \overline{W}}$  is singular at  $W = 0$ , both  $\cos \sqrt{W \cdot \overline{W}}$  and  $\sqrt{W \cdot \overline{W}} \sin t\sqrt{W \cdot \overline{W}}$  are holomorphic. The Grauert tube function equals

$$\sqrt{\rho}(z) = i \cosh^{-1} |z|^2, \quad (z \in \mathbb{S}_{\mathbb{C}}^n).$$

It is globally well defined on  $\mathbb{S}_{\mathbb{C}}^n$ . The characteristic conoid is defined by  $\cosh \frac{1}{i}\sqrt{\rho} = \cosh \tau$ .

(iii).  $\mathbb{H}^n$  The hyperboloid model of hyperbolic space is the hypersurface in  $\mathbb{R}^{n+1}$  defined by

$$\mathbb{H}^n = \{x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1, \quad x_n > 0\}.$$

Then,

$$H_{\mathbb{C}}^n = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : z_1^2 + \cdots + z_n^2 - z_{n+1}^2 = -1\}.$$

In real coordinates  $z_j = x_j + i\xi_j$ , this is:

$$\langle x, x \rangle_L - \langle \xi, \xi \rangle_L = -1, \quad \langle x, \xi \rangle_L = 0$$

where  $\langle \cdot, \cdot \rangle_L$  is the Lorentz inner product of signature  $(n, 1)$ . Hence the complexified hyperbolic space is the hypersurface in  $\mathbb{C}^{n+1}$  given by the same equations.

We obtain  $\mathbb{H}_{\mathbb{C}}^n$  from  $\mathbb{S}_{\mathbb{C}}^n$  by the map  $(z', z_{n+1}) \rightarrow (iz', z_{n+1})$ . The complexified geodesic flow is given for  $((Z, W) \in T^*\mathbb{H}^n)$ . by

$$\begin{aligned} G^t(Z, W) &= (\cosh t\sqrt{\langle W, W \rangle_L}Z + (\sinh t\sqrt{\langle W, W \rangle_L})\frac{W}{\sqrt{\langle W, W \rangle_L}}), \\ &\quad -\sqrt{\langle W, W \rangle_L}(\sinh t\sqrt{\langle W, W \rangle_L})Z + (\cosh t\sqrt{\langle W, W \rangle_L})W. \end{aligned}$$

The Grauert tube function is:

$$\sqrt{\rho}(z) = \cos^{-1}(\|x\|_L^2 + \|\xi\|_L^2 - \pi)/\sqrt{2}.$$

The radius of maximal Grauert tube is  $\varepsilon = 1$  or  $r = \pi/\sqrt{2}$ .

ACGLOBAL

## 3. ANALYTIC CONTINUATION OF EIGENFUNCTIONS

A function  $f$  on a real analytic manifold  $M$  is real analytic,  $f \in C^\omega(M)$ , if and only if it satisfies the Cauchy estimates

$$(3.1) \quad |D^\alpha f(x)| \leq K L^{|\alpha|} \alpha!$$

for some  $K, L > 0$ . In place of all derivatives it is sufficient to use powers of  $\Delta$ . In the language of Baouendi-Goulaouic [BG, BG2, BG3], the Laplacian of a compact real analytic Riemannian manifold has the property of iterates, i.e. the real analytic functions are precisely the functions satisfying Cauchy estimates relative to  $\Delta$ ,

$$(3.2) \quad C^\omega(M) = \{u \in C^\infty(M) : \exists L > 0, \forall k \in \mathbf{N}, \|\Delta^k u\|_{L^2(M)} \leq L^{k+1} (2k)!\}.$$

It is classical that all of the eigenfunctions extend holomorphic to a fixed Grauert tube.

**Theorem 3.1.** (*Morrey-Nirenberg Theorem*) *Let  $P(x, D)$  be an elliptic differential operator in  $\Omega$  with coefficients which are analytic in  $\Omega$ . If  $u \in \mathcal{D}'(\Omega)$  and  $P(x, D)u = f$  with  $f \in C^\omega(\Omega)$ , then  $u \in C^\omega(\Omega)$ .*

The proof shows that the radius of convergence of the solution is determined by the radius of convergence of the coefficients.

Let us consider examples of holomorphic continuations of eigenfunctions:

- On the flat torus  $\mathbb{R}^m/\mathbb{Z}^m$ , the real eigenfunctions are  $\cos\langle k, x \rangle, \sin\langle k, x \rangle$  with  $k \in 2\pi\mathbb{Z}^m$ . The complexified torus is  $\mathbb{C}^m/\mathbb{Z}^m$  and the complexified eigenfunctions are  $\cos\langle k, \zeta \rangle, \sin\langle k, \zeta \rangle$  with  $\zeta = x + i\xi$ .
- On the unit sphere  $S^m$ , eigenfunctions are restrictions of homogeneous harmonic functions on  $\mathbb{R}^{m+1}$ . The latter extend holomorphically to holomorphic harmonic polynomials on  $\mathbb{C}^{m+1}$  and restrict to holomorphic function on  $S_{\mathbb{C}}^m$ .
- On  $\mathbf{H}^m$ , one may use the hyperbolic plane waves  $e^{(i\lambda+1)\langle z, b \rangle}$ , where  $\langle z, b \rangle$  is the (signed) hyperbolic distance of the horocycle passing through  $z$  and  $b$  to 0. They may be holomorphically extended to the maximal tube of radius  $\pi/4$ .
- On compact hyperbolic quotients  $\mathbf{H}^m/\Gamma$ , eigenfunctions can be then represented by Helgason's generalized Poisson integral formula [H],

$$\varphi_\lambda(z) = \int_B e^{(i\lambda+1)\langle z, b \rangle} dT_\lambda(b).$$

Here,  $z \in D$  (the unit disc),  $B = \partial D$ , and  $dT_\lambda \in \mathcal{D}'(B)$  is the boundary value of  $\varphi_\lambda$ , taken in a weak sense along circles centered at the origin 0. To analytically continue  $\varphi_\lambda$  it suffices to analytically continue  $\langle z, b \rangle$ . Writing the latter as  $\langle \zeta, b \rangle$ , we have:

$$(3.3) \quad \varphi_\lambda^{\mathbb{C}}(\zeta) = \int_B e^{(i\lambda+1)\langle \zeta, b \rangle} dT_\lambda(b).$$

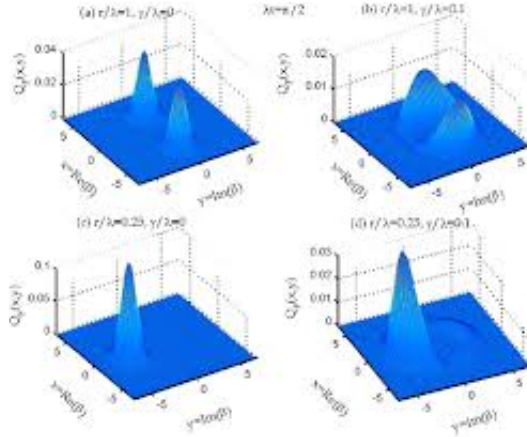
In Theorem 2 of [BG2] and Theorem 1.2 of [BGH] it is proved that the operator  $\Delta$  has the iterate property if and only if, for all  $b > 1$ , each eigenfunction extends holomorphically to some Grauert tube  $M_\tau$  and satisfies

$$(3.4) \quad \sup_{z \in M_\tau} |\varphi_{\lambda_j}^{\mathbb{C}}(z)| \leq b^{\lambda_j} \sup_{x \in M} |\varphi_{\lambda_j}(x)|.$$

BGEST

The concept of Grauert was not actually used in these articles, so the relation between the growth rate and the Grauert tube function was not stated. But it again shows that all eigenfunctions extend to some fixed Grauert tube.

1. **Husimi functions.** The ( $L^2$ -normalizations of the) modulus squares (1.2) are sometimes known as Husimi functions (after [Hu40]). They are holomorphic extensions of  $L^2$ -normalized functions but are not themselves  $L^2$  normalized on  $M_\varepsilon$ . However, as will be discussed below, their  $L^2$  norms may on the Grauert tubes (and their boundaries) can be determined. One can then ask how the mass of the normalized Husimi function is distributed in phase space, or how the  $L^p$  norms behave.



One of the general problems of quantum dynamics is to determine all of the weak\* limits of the sequence,

$$\left\{ \frac{|\varphi_j^C(z)|^2}{\|\varphi_j^C\|_{L^2(\partial M_\varepsilon)}} d\mu_\varepsilon \right\}_{j=1}^\infty.$$

Here,  $d\mu_\varepsilon$  is the natural measure on  $\partial M_\varepsilon$  corresponding to the contact volume form on  $S_\varepsilon^*M$ . Recall that a sequence  $\mu_n$  of probability measures on a compact space  $X$  is said to converge weak\* to a measure  $\mu$  if  $\int_X f d\mu_n \rightarrow \int_X f d\mu$  for all  $f \in C(X)$ . We refer to Theorem [??] for the ergodic case. In the integrable case one has localization results showing that complex zeros lie on hypersurfaces (see [ZeZ]).

#### 4. POISSON-WAVE OPERATOR AND SZEGÖ PROJECTOR ON GRAUERT TUBES

In this section, we introduce the Poisson-wave operator, the Szegő projector, and complexified spectral projections and state some basic results on analytic continuation and growth (Theorem 4.1 and Theorem 4.3). The theorems on analytic continuation of the Poisson wave kernel are proved in Section 5 following [Ze12, L13]. The theorems on growth of complexified and tempered spectral projections are proved in Section 6 with refinements sketched in Section [??].

1. **Poisson operator and analytic Continuation of eigenfunctions.** The half-wave group of  $(M, g)$  is the unitary group  $U(t) = e^{it\sqrt{\Delta}}$  generated by the square root of the

positive Laplacian. Its Schwartz kernel is a distribution on  $\mathbb{R} \times M \times M$  with the eigenfunction expansion

$$\boxed{\text{Ut}} \quad (4.1) \quad U(t, x, y) = \sum_{j=0}^{\infty} e^{it\lambda_j} \varphi_j(x) \varphi_j(y).$$

By the Poisson operator we mean the analytic continuation of  $U(t)$  to positive imaginary time,

$$\boxed{\text{POISSON}} \quad (4.2) \quad e^{-\tau\sqrt{\Delta}} = U(i\tau).$$

The eigenfunction expansion then converges absolutely to a real analytic function on  $\mathbb{R}_+ \times M \times M$ .

Let  $A(\tau)$  denote the operator of analytic continuation of a function on  $M$  to the Grauert tube  $M_\tau$ . Since

$$\boxed{\text{ACEFN}} \quad (4.3) \quad U_{\mathbb{C}}(i\tau) \varphi_\lambda = e^{-\tau\lambda} \varphi_\lambda^{\mathbb{C}},$$

it is simple to see that

$$\boxed{\text{ATAU}} \quad (4.4) \quad A(\tau) = U_{\mathbb{C}}(i\tau) e^{\tau\sqrt{\Delta}}$$

where  $U_{\mathbb{C}}(i\tau, \zeta, y)$  is the analytic continuation of the Poisson kernel in  $x$  to  $M_\tau$ . In terms of the eigenfunction expansion, one has

$$\boxed{\text{UI}} \quad (4.5) \quad U(i\tau, \zeta, y) = \sum_{j=0}^{\infty} e^{-\tau\lambda_j} \varphi_j^{\mathbb{C}}(\zeta) \varphi_j(y), \quad (\zeta, y) \in M_\varepsilon \times M.$$

This is a very useful observation because  $U_{\mathbb{C}}(i\tau) e^{\tau\sqrt{\Delta}}$  is a Fourier integral operator with complex phase and can be related to the geodesic flow. The analytic continuability of the Poisson operator to  $M_\tau$  implies that every eigenfunction analytically continues to the same Grauert tube.

**2. Analytic continuation of the Poisson wave group.** The analytic continuation of the Poisson-wave kernel to  $M_\tau$  in the  $x$  variable is discussed in detail in [Ze12] and ultimately derives from the analysis by Hadamard of his parametrix construction. We only briefly discuss it here and refer to [Ze12] for further details. In the case of Euclidean  $\mathbb{R}^n$  and its wave kernel  $U(t, x, y) = \int_{\mathbb{R}^n} e^{it|\xi|} e^{i\langle \xi, x-y \rangle} d\xi$  which analytically continues to  $t + i\tau, \zeta = x + ip \in \mathbb{C}_+ \times \mathbb{C}^n$  as the integral

$$U_{\mathbb{C}}(t + i\tau, x + ip, y) = \int_{\mathbb{R}^n} e^{i(t+i\tau)|\xi|} e^{i\langle \xi, x+ip-y \rangle} d\xi.$$

The integral clearly converges absolutely for  $|p| < \tau$ .

Exact formulae of this kind exist for  $S^m$  and  $\mathbf{H}^m$ . For a general real analytic Riemannian manifold, there exists an oscillatory integral expression for the wave kernel of the form,

$$\boxed{\text{PARAONE}} \quad (4.6) \quad U(t, x, y) = \int_{T_y^* M} e^{it|\xi|_{g_y}} e^{i\langle \xi, \exp_y^{-1}(x) \rangle} A(t, x, y, \xi) d\xi$$

where  $A(t, x, y, \xi)$  is a polyhomogeneous amplitude of order 0. The holomorphic extension of (4.6) to the Grauert tube  $|\zeta| < \tau$  in  $x$  at time  $t = i\tau$  then has the form

$$\text{CXPARAONE} \quad (4.7) \quad U_{\mathbb{C}}(i\tau, \zeta, y) = \int_{T_y^*} e^{-\tau|\xi|_{g_y}} e^{i(\xi, \exp_y^{-1}(\zeta))} A(t, \zeta, y, \xi) d\xi \quad (\zeta = x + ip).$$

**3. Complexified spectral projections.** The next step is to holomorphically extend the spectral projectors  $d\Pi_{[0, \lambda]}(x, y) = \sum_j \delta(\lambda - \lambda_j) \varphi_j(x) \varphi_j(y)$  of  $\sqrt{\Delta}$ . The complexified diagonal spectral projections measure is defined by

$$(4.8) \quad d_{\lambda} \Pi_{[0, \lambda]}^{\mathbb{C}}(\zeta, \bar{\zeta}) = \sum_j \delta(\lambda - \lambda_j) |\varphi_j^{\mathbb{C}}(\zeta)|^2.$$

Henceforth, we generally omit the superscript and write the kernel as  $\Pi_{[0, \lambda]}^{\mathbb{C}}(\zeta, \bar{\zeta})$ . This kernel is not a tempered distribution due to the exponential growth of  $|\varphi_j^{\mathbb{C}}(\zeta)|^2$ . Since many asymptotic techniques assume spectral functions are of polynomial growth, we simultaneously consider the damped spectral projections measure

$$\text{PROJDAMPED} \quad (4.9) \quad d_{\lambda} P_{[0, \lambda]}^{\tau}(\zeta, \bar{\zeta}) = \sum_j \delta(\lambda - \lambda_j) e^{-2\tau\lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2,$$

which is a temperate distribution as long as  $\sqrt{\rho}(\zeta) \leq \tau$ . When we set  $\tau = \sqrt{\rho}(\zeta)$  we omit the  $\tau$  and put

$$\text{PROJDAMPEDz} \quad (4.10) \quad d_{\lambda} P_{[0, \lambda]}(\zeta, \bar{\zeta}) = \sum_j \delta(\lambda - \lambda_j) e^{-2\sqrt{\rho}(\zeta)\lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2.$$

The integral of the spectral measure over an interval  $I$  gives

$$\Pi_I(x, y) = \sum_{j: \lambda_j \in I} \varphi_j(x) \varphi_j(y).$$

Its complexification gives the spectral projections kernel along the anti-diagonal,

$$\text{CXSP} \quad (4.11) \quad \Pi_I(\zeta, \bar{\zeta}) = \sum_{j: \lambda_j \in I} |\varphi_j^{\mathbb{C}}(\zeta)|^2,$$

and the integral of (4.9) gives its temperate version

$$\text{CXDSP} \quad (4.12) \quad P_I^{\tau}(\zeta, \bar{\zeta}) = \sum_{j: \lambda_j \in I} e^{-2\tau\lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2,$$

or in the crucial case of  $\tau = \sqrt{\rho}(\zeta)$ ,

$$\text{CXDSPa} \quad (4.13) \quad P_I(\zeta, \bar{\zeta}) = \sum_{j: \lambda_j \in I} e^{-2\sqrt{\rho}(\zeta)\lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2,$$

4. **Poisson operator as a complex Fourier integral operator.** The damped spectral projection measure  $d_\lambda P_{[0,\lambda]}^\tau(\zeta, \bar{\zeta})$  (4.9) is dual under the real Fourier transform in the  $t$  variable to the restriction

$$(4.14) \quad U(t + 2i\tau, \zeta, \bar{\zeta}) = \sum_j e^{(-2\tau+it)\lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2$$

to the anti-diagonal of the mixed Poisson-wave group. The adjoint of the Poisson kernel  $U(i\tau, x, y)$  also admits an anti-holomorphic extension in the  $y$  variable. The sum (4.14) are the diagonal values of the complexified wave kernel

$$(4.15) \quad \begin{aligned} U(t + 2i\tau, \zeta, \bar{\zeta}') &= \int_M U(t + i\tau, \zeta, y) E(i\tau, y, \bar{\zeta}') dV_g(x) \\ &= \sum_j e^{(-2\tau+it)\lambda_j} \varphi_j^{\mathbb{C}}(\zeta) \overline{\varphi_j^{\mathbb{C}}(\zeta')} \end{aligned}$$

We obtain (4.15) by orthogonality of the real eigenfunctions on  $M$ .

Since  $U(t + 2i\tau, \zeta, y)$  takes its values in the CR holomorphic functions on  $\partial M_\tau$ , we consider the Sobolev spaces  $\mathcal{O}^{s+\frac{n-1}{4}}(\partial M_\tau)$  of CR holomorphic functions on the boundaries of the strictly pseudo-convex domains  $M_\varepsilon$ , i.e.

$$\mathcal{O}^{s+\frac{m-1}{4}}(\partial M_\tau) = W^{s+\frac{m-1}{4}}(\partial M_\tau) \cap \mathcal{O}(\partial M_\tau),$$

where  $W_s$  is the  $s$ th Sobolev space and where  $\mathcal{O}(\partial M_\varepsilon)$  is the space of boundary values of holomorphic functions. The inner product on  $\mathcal{O}^0(\partial M_\tau)$  is with respect to the Liouville measure

$$(4.16) \quad d\mu_\tau = (i\partial\bar{\partial}\sqrt{\rho})^{m-1} \wedge d^c\sqrt{\rho}.$$

We then regard  $U(t + i\tau, \zeta, y)$  as the kernel of an operator from  $L^2(M) \rightarrow \mathcal{O}^0(\partial M_\tau)$ . It equals its composition  $\Pi_\tau \circ U(t + i\tau)$  with the Szegö projector

$$\Pi_\tau : L^2(\partial M_\tau) \rightarrow \mathcal{O}^0(\partial M_\tau)$$

for the tube  $M_\tau$ , i.e. the orthogonal projection onto boundary values of holomorphic functions in the tube.

This is a useful expression for the complexified wave kernel, because  $\tilde{\Pi}_\tau$  is a complex Fourier integral operator with a small wave front relation. More precisely, the real points of its canonical relation form the graph  $\Delta_\Sigma$  of the identity map on the symplectic one  $\Sigma_\tau \subset T^*\partial M_\tau$  spanned by the real one-form  $d^c\rho$ , i.e.

$$(4.17) \quad \Sigma_\tau = \{(\zeta; rd^c\rho(\zeta)), \quad \zeta \in \partial M_\tau, r > 0\} \subset T^*(\partial M_\tau).$$

We note that for each  $\tau$ , there exists a symplectic equivalence  $\Sigma_\tau \xrightarrow{\sim} T^*M$  by the map  $(\zeta, rd^c\rho(\zeta)) \rightarrow (E_{\mathbb{C}}^{-1}(\zeta), r\alpha)$ , where  $\alpha = \xi \cdot dx$  is the action form (cf. [GS2]).

The following result was first stated by Boutet de Monvel [Bou] and has been proved in detail in [Ze12, L13, Ste].

**Theorem 4.1.**  $\Pi_\varepsilon \circ U(i\varepsilon) : L^2(M) \rightarrow \mathcal{O}(\partial M_\varepsilon)$  is a complex Fourier integral operator of order  $-\frac{m-1}{4}$  associated to the canonical relation

$$\Gamma = \{(y, \eta, \iota_\varepsilon(y, \eta))\} \subset T^*M \times \Sigma_\varepsilon.$$

Moreover, for any  $s$ ,

$$\Pi_\varepsilon \circ U(i\varepsilon) : W^s(M) \rightarrow \mathcal{O}^{s+\frac{m-1}{4}}(\partial M_\varepsilon)$$

is a continuous isomorphism.

**5. Complexified Poisson kernel as a complex Fourier integral operator.** The following theorem is stated in [Bou] (For proofs, see [Ze12, LI3]):

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**Theorem 4.2.** (see [Bou, GS2, GLS]) For sufficiently small  $\tau > 0$ ,  $U_{\mathbb{C}}(i\tau) : L^2(M) \rightarrow \mathcal{O}(\partial M_{\tau})$  is a Fourier integral operator of order  $-\frac{m-1}{4}$  with complex phase associated to the canonical relation

$$\Lambda = \{(y, \eta, \iota_{\tau}(y, \eta))\} \subset T^*M \times \Sigma_{\tau}.$$

Moreover, for any  $s$ ,

$$U_{\mathbb{C}}(i\tau) : W^s(M) \rightarrow \mathcal{O}^{s+\frac{m-1}{4}}(\partial M_{\tau})$$

is a continuous isomorphism.

The proof of Theorem 4.2 is barely sketched in [Bou]. However, the theorem follows almost immediately from the construction of the branched meromorphic Hadamard parametrix in Corollary 5.1, or alternatively from the analytic continuation of the Hörmander parametrix of §4. It suffices to show that either is a parametrix for  $U_{\mathbb{C}}(i\tau, \zeta, y)$ , i.e. differs from it by an analytic kernel (smooth would be sufficient by analytic wave front set considerations). But the Hadamard parametrix construction is an exact formula and actually gives a more precise description of the singularities of  $U_{\mathbb{C}}(i\tau, \zeta, y)$  than is stated in Theorem 4.2. We briefly explain how either the Hadamard or Hörmander parametrix can be used to complete the proof.

Using the complexified Poisson-wave kernel, one can prove the following sup-norm estimate:

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**Proposition 4.3.** Suppose  $(M, g)$  is real analytic. Then

$$\sup_{\zeta \in M_{\tau}} |\varphi_{\lambda}^{\mathbb{C}}(\zeta)| \leq C\lambda^{\frac{m+1}{2}} e^{\tau\lambda}, \quad \sup_{\zeta \in M_{\tau}} \left| \frac{\partial \varphi_{\lambda}^{\mathbb{C}}(\zeta)}{\partial \zeta_j} \right| \leq C\lambda^{\frac{m+3}{2}} e^{\tau\lambda}$$

ACPW

## 5. ANALYTIC CONTINUATION OF THE POISSON-WAVE KERNEL

In this section we prove Theorem 4.1 and Theorem 4.2, closely following [Ze12]. Other closely related proofs can be found in [LI3, Ste].

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**1. Hörmander parametrix for the Poisson-wave kernel.** A more familiar construction of  $U(t, x, y)$  and its analytic continuation which is particularly useful for small  $|t|$  is the one based on the Fourier inversion formula. Its generalization to Riemannian manifolds is given by

$$(5.1) \quad U(t, x, y) = \int_{T_y^*M} e^{it|\xi|_{g_y}} e^{i\langle \xi, \exp_y^{-1}(x) \rangle} A(t, x, y, \xi) d\xi,$$

for  $(x, y)$  sufficiently close to the diagonal. We use this parametrix to prove Theorem 6.5 (2).

The amplitude is a polyhomogeneous symbol of the form

AMP

$$(5.2) \quad A(t, x, y, \xi) \sim \sum_{j=0}^{\infty} A_j(t, x, y, \xi),$$

where the asymptotics are in the sense of the symbol topology and where

$$A_j(t, x, y, \tau\xi) = \tau^{-j} A_j(t, x, y, \xi), \quad \text{for } |\xi| \geq 1.$$

The principal term  $A_0(t, x, y, \xi)$  equals 1 when  $t = 0$  on the diagonal, and the higher  $A_j$  are determined by transport equations discussed in [DG].

It can be verified that in the case of real analytic  $(M, g)$ , the amplitude is a classical formal analytic symbol (see §8). Hence if  $\mathcal{A}(t, x, y, \xi)$  is a realization of the amplitude  $A(t, x, y, \xi)$ , then one obtains an analytic parametrix

**PARAONEan** (5.3) 
$$U(t, x, y) = \int_{T_y^*M} e^{it|\xi|_{g_y}} e^{i\langle \xi, \exp_y^{-1}(x) \rangle} \mathcal{A}(t, x, y, \xi) d\xi,$$

which approximates the wave kernel for small  $|t|$  and  $(x, y)$  near the diagonal up to a holomorphic error, whose amplitude is exponentially decaying in  $|\xi|$ .

**2. Fourier integral distributions with complex phase.** First, we review the relevant definitions (see [HoIV], §25.5 or [MeSj]). A Fourier integral distribution with complex phase on a manifold  $X$  is a distribution that can locally be represented by an oscillatory integral

$$A(x) = \int_{\mathbb{R}^N} e^{i\varphi(x, \theta)} a(x, \theta) d\theta$$

where  $a(x, \theta) \in S^m(X \times V)$  is a symbol of order  $m$  in a cone  $V \subset \mathbb{R}^N$  and where the phase  $\varphi$  is a positive regular phase function, i.e. it satisfies

- $\text{Im } \varphi \geq 0$ ;
- $d \frac{\partial \varphi}{\partial \theta_1}, \dots, d \frac{\partial \varphi}{\partial \theta_N}$  are linearly independent complex vectors on

$$C_{\varphi\mathbb{R}} = \{(x, \theta) : d_\theta \varphi(x, \theta) = 0\}.$$

- In the analytic setting (which is assumed in this article),  $\varphi$  admits an analytic continuation  $\varphi_{\mathbb{C}}$  to an open cone in  $X_{\mathbb{C}} \times V_{\mathbb{C}}$ .

Define

$$C_{\varphi_{\mathbb{C}}} = \{(x, \theta) \in X_{\mathbb{C}} \times V_{\mathbb{C}} : \nabla_\theta \varphi_{\mathbb{C}}(x, \theta) = 0\}.$$

Then  $C_{\varphi_{\mathbb{C}}}$  is a manifold near the real domain. One defines the Lagrangian submanifold  $\Lambda_{\varphi_{\mathbb{C}}} \subset T^*X_{\mathbb{C}}$  as the image

$$(x, \theta) \in C_{\varphi_{\mathbb{C}}} \rightarrow (x, \nabla_x \varphi_{\mathbb{C}}(x, \theta)).$$

**AC1** **3. Analytic continuation of the Hadamard parametrix.** As in §8 and §??, we can express  $U_{\mathbb{C}}(i\tau, \zeta, y)$  as a local Fourier integral distribution with complex phase by rewriting the Hadamard series in Corollary 5.1 as oscillatory integrals. Here we assume that  $\tau > 0, t \geq 0$ .

A complication is that we can only use the complexified phase  $\Gamma = \frac{t^2}{2} - r^2$  in regions of complexified  $\mathbb{R} \times M \times M$  where its imaginary part is  $\geq 0$ . As in §??, we could also use the phase  $t - r$  (resp.  $t + r$ ) in regions where  $t + r \neq 0$  (resp.  $t - r \neq 0$ ) and where the contour  $\mathbb{R}_+$  can be deformed back to itself after the the change of variables  $\theta \rightarrow (t + r)\theta$ .

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4. **Analytic continuation of the Hörmander parametrix.** As was the case in  $\mathbb{R}^n$ , the parametrix (5.3) admits an analytic continuation in time to a strip  $\{t + i\tau : \tau < \tau_{an}, |t| < 1\}$ . In the space variables, the parametrix then admits an analytic continuation to complex  $x, y$  satisfying  $|r_{\mathbb{C}}(x, y)| \leq \tau$ .

The analytically continued parametrix (4.6) approximates the true analytically continued Poisson kernel up to a holomorphic kernel. More precisely, for any  $x_0 \in M$  and  $\tau > 0$ , there exists  $\varepsilon, \rho > 0$  and an open neighborhood  $W$  of  $x_0$  in  $M_\tau$  such that for  $|t| < 1$  and  $(x, y) \in W \times W$ ,

$$(5.4) \quad U(t + i\tau, x, y) = \int_{T_y^*M} e^{-\tau|\xi|_{g_y}} e^{i(\xi, \exp_y^{-1}(x))} \mathcal{A}(t + i\tau, x, y, \xi) d\xi + R(t, x, y),$$

where  $R(t, x, y)$  is holomorphic for small  $|t|$  and for  $(x, y)$  near the diagonal.

The parametrix is only defined near the diagonal where  $\exp_y^{-1}$  is defined. However one can extend it to a global holomorphic kernel away from  $\mathcal{C}_{\mathbb{C}}$  by cutting off the first term of (4.6) with a smooth cutoff  $\chi(x, y)$  supported near the diagonal in  $M_\tau \times M_\tau$  and then solving a  $\bar{\partial}$  problem on the Grauert tube (or a  $\bar{\partial}_b$  problem on its boundary) to extend the kernel to be globally holomorphic (resp. CR). We refer to [Ze12] for a more detailed discussion. This gives an alternative to the Hadamard parametrix construction of Corollary 5.1.

This concludes the sketch of proof of Theorem 4.2.

5.  **$\Delta_g, \square_g$  and characteristics.** In the real domain,  $\Delta$  is an elliptic operator with principal symbol  $\sigma_\Delta(x, \xi) =: \sum_{i,j=1}^n g^{ij}(x) \xi_i \xi_j$ . Hence its characteristic set (the zero set of its symbol) consists only of the zero section  $\xi = 0$  in  $T^*M$ . But when we continue it to the complex domain it develops a complex characteristic set

$$(5.5) \quad \text{Ch}(\Delta_{\mathbb{C}}) = \{(\zeta, \xi) \in T^*M_{\mathbb{C}} : \sum_{i,j=1}^n g^{ij}(\zeta) \xi_i \xi_j = 0\}.$$

The wave operator on the product spacetime  $(\mathbb{R} \times M, dt^2 - g_x)$  is given by

$$\square_g = \frac{\partial^2}{\partial t^2} + \Delta_g.$$

The unusual sign in front of  $\Delta_g$  is due to the sign normalization above making the Laplacian non-negative. Again we omit the subscript when the metric is fixed. The characteristic variety of  $\square$  is the zero set of its symbol

$$\sigma_{\square}(t, \tau, x, \xi) = \tau^2 - |\xi|_x^2,$$

that is,

$$(5.6) \quad \text{Ch}(\square) = \{(t, \tau, x, \xi) \in T^*(\mathbb{R} \times M) : \tau^2 - |\xi|_x^2 = 0\}.$$

The null-bicharacteristic flow of  $\square$  is the Hamiltonian flow of  $\tau^2 - |\xi|_x^2$  on  $\text{Ch}(\square)$ . Its graph is thus

$$\Lambda = \{(t, \tau, x, \xi, y, \eta) : \tau^2 - |\xi|_x^2 = 0, G^t(x, \xi) = (y, \eta)\} \subset T^*(\mathbb{R} \times M \times M).$$

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6. **Characteristic variety and characteristic conoid.** Following [H], we put

$$(5.7) \quad \Gamma(t, x, y) = t^2 - r^2(x, y).$$

Here,  $r(x, y)$  is the distance between  $x, y$ . It is singular at  $r = 0$  and also when  $y$  is in the ‘cut locus’ of  $x$ . In this article we only consider  $(x, y)$  so that  $r(x, y) < \text{inj}(x)$ , where  $\text{inj}(x)$  is the injectivity radius at  $x$ , i.e. is the largest  $\varepsilon$  so that

$$\exp_x : B_{x, \varepsilon}^* M \rightarrow M$$

is a diffeomorphism to its image. The injectivity radius  $\text{inj}(M, g)$  is the maximum of  $\text{inj}(x)$  for  $x \in M$ . Thus, we work in a sufficiently small neighborhood of the diagonal so that cut points do not occur.

The squared distance  $r^2(x, y)$  is smooth in a neighborhood of the diagonal. On a simply connected manifold  $(\tilde{M}, g)$  without conjugate points, it is globally smooth on  $\tilde{M} \times \tilde{M}$ . We recall that ‘without conjugate points’ means that  $\exp_x : T_x M \rightarrow M$  is non-singular for all  $x$ .

The characteristic conoid is the set

$$(5.8) \quad \mathcal{C} = \{(t, x, y) : r(x, y) < \text{inj}(x), r^2(x, y) = t^2\} \subset \mathbb{R} \times M \times M.$$

It separates  $\mathbb{R} \times M \times M$  into the forward/backward semi-cones

$$\mathcal{C}_{\pm} = \{(t, x, y) : t^2 - r^2(x, y) > 0, \pm t > 0\}.$$

The complexification of  $\mathcal{C} = \mathcal{C}_{\mathbb{R}}$  is the complex characteristic conoid

$$(5.9) \quad \mathcal{C}_{\mathbb{C}} = \{(t, x, y) : r_{\mathbb{C}}^2(x, y) = t^2\} \subset \mathbb{C} \times M_{\mathbb{C}} \times M_{\mathbb{C}}.$$

We note that  $\mathcal{C}_{\mathbb{R}} \subset \mathcal{C}_{\mathbb{C}}$  is a totally real submanifold. Another totally real submanifold of central importance in this article is the ‘diagonal’ (or anti-diagonal) conoid,

$$(5.10) \quad \mathcal{C}_{\Delta} = \{(2i\tau, \zeta, \bar{\zeta}) : \tau \in \mathbb{R}_+, \zeta, \bar{\zeta} \in \partial M_{\tau}\}.$$

By definition,  $r_{\mathbb{C}}^2(\zeta, \bar{\zeta}) = -4\tau^2$  if  $\zeta \in \partial M_{\tau}$ .

7. **Hadamard parametrix for the Poisson-wave kernel.** We are most interested in the Hadamard parametrix for the half-wave kernel, which does not seem to have been discussed in the literature. We are more generally interested in the Poisson-wave semi-group  $e^{i(t+i\tau)\sqrt{\Delta}}$  for  $\tau > 0$ . The Poisson-wave kernel

$$(5.11) \quad U(t + i\tau, x, y) = \sum_j e^{i(t+i\tau)\lambda_j} \varphi_j(x) \varphi_j(y)$$

is a real analytic kernel which possesses an analytic extension to a Grauert tube. Thus, there exists a non-zero analytic radius  $\tau_{an} > 0$  so that the Poisson kernel admits a holomorphic extension  $U(t + i\tau, \zeta, y)$  to  $M_{\tau} \times M$  for  $\tau \leq \tau_{an}$ . Since

$$(5.12) \quad U(i\tau) \varphi_{\lambda} = e^{-\tau\lambda} \varphi_{\lambda}^{\mathbb{C}},$$

the eigenfunctions analytically extend to the same maximal tube as does  $U(i\tau)$ .

We would like to construct a Hadamard type parametrix for (5.11). We may derive it from the Feynman-Hadamard fundamental solution using that

$$(5.13) \quad \frac{d}{dt} \frac{e^{i|t|\sqrt{\Delta}}}{\sqrt{\Delta}} = i \text{sgn}(t) e^{i|t|\sqrt{\Delta}}$$

and

$$(5.14) \quad e^{it\sqrt{\Delta}} = \frac{1}{i}H(t)\frac{d}{dt}\frac{e^{i|t|\sqrt{\Delta}}}{\sqrt{\Delta}} - \frac{1}{i}H(-t)\frac{d}{dt}\frac{e^{-i|t|\sqrt{\Delta}}}{\sqrt{\Delta}}.$$

Hence,

$$\boxed{\text{ddt}} \quad (5.15) \quad \frac{d}{idt}U_F(t) = e^{it\sqrt{\Delta}}, \quad (t > 0).$$

The restriction to  $t > 0$  is consistent with the fact that  $e^{it\sqrt{\Delta}}(x, y)$  has the singularity  $((t + i0)^2 - r^2)^{-\frac{m}{2}}$  (in odd spacetime dimensions) while  $U_F(t)$  has the singularity  $(t^2 - r^2 + i0)^{\frac{2-m}{2}}$ . We note (again) that  $((t + i0)^2 - r^2)^\alpha = (t^2 - r^2 + i0)^\alpha$  for  $t > 0$ .

From Theorem 7.7 we conclude:

$\boxed{\text{HALFHAD}}$  **Corollary 5.1.** *Let  $(M, g)$  be real analytic. Then with the  $U_j, V_k, W_\ell$  defined as in Theorem 5.1, we have:*

- In odd spacetime dimensions, for  $t > 0$  the Poisson-wave kernel  $U(t + i\tau, x, y)$  ( $\tau > 0$ ) has the form  $A\Gamma^{-\frac{m}{2}}$  where  $A = \sum_{j=0}^{\infty} A_j\Gamma^j$  with  $A_j$  holomorphic. The series converges absolutely to a holomorphic function for  $|\Gamma| < \varepsilon$  sufficient small, i.e. near the characteristic conoid.
- In even spacetime dimensions, for  $t > 0$ , the Poisson-wave kernel has the form  $B\Gamma^{-\frac{m}{2}} + C \log \Gamma + D$  where the coefficients  $B, C, D$  are holomorphic in a neighborhood of  $\mathcal{C}_{\mathbb{C}}$ , and have the same  $\Gamma$  expansions as  $A$ .

We use this parametrix to prove Theorem 6.5 (I).

$\boxed{\text{OSCHAD}}$  8. **Hadamard parametrix as an oscillatory integral with complex phase.** Corollary 5.1 gives a precise description of the singularities of the Poisson-wave propagator. It implicitly describes the kernel as a Fourier integral kernel. We now make this description explicit in the real domain. In the following sections, we extend the description to the complex domain.

We first express  $\Gamma^{-\frac{m}{2}+j}$  as an oscillatory integral with one phase variable using the well-known identity

$$\boxed{\text{INT}} \quad (5.16) \quad \int_0^{\infty} e^{i\theta\sigma}\theta_+^\lambda d\lambda = ie^{i\lambda\pi/2}\Gamma(\lambda + 1)(\sigma + i0)^{-\lambda-1}.$$

At least formally, this leads to the representation

$$\int_0^{\infty} e^{i\theta(t^2-r^2)}\theta_+^{\frac{n-1}{2}-j} d\theta = ie^{i(\frac{n-1}{2}-j)\pi/2}\Gamma(\frac{n-1}{2} - j + 1)(t^2 - r^2 + i0)^{j-\frac{n-1}{2}-1}$$

for the principal term of the Poisson-wave. Here, the notation  $\Gamma = t^2 - r^2$  unfortunately clashes with that for the Gamma function, and we temporarily write out its definition.

In even space dimensions, the Hadamard parametrix for the Hadamard-Feynman fundamental solution thus has the form

$$\boxed{\text{SUMUJ}} \quad (5.17) \quad \sum_{j=0}^{\infty} U_j(t, x, y)\Gamma^{\frac{1-n}{2}+j} = \int_0^{\infty} e^{i\theta(t^2-r^2)} \left( \sum_{j=0}^{\infty} U_j(t, x, y) (ie^{i(\frac{n-1}{2}-j)\pi/2})^{-1} \frac{\theta_+^{\frac{n-3}{2}-j}}{\Gamma(\frac{n-3}{2}-j+1)} \right) d\theta.$$

Here we follow Hadamard's notation, but it is simpler to re-define the coefficients  $U_j$  so that the  $\Gamma$ -factors appear on the left side as in [Be] (7). We thus define

$$\mathcal{U}_j(t, x, y) = \left( (ie^{i(\frac{n-1}{2}-j)\pi/2})^{-1} \frac{1}{\Gamma(\frac{n-3}{2} - j + 1)} \right) U_j(t, x, y).$$

By the duplication formula  $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$  with  $z = \frac{m}{2} - k - \frac{\alpha}{2}$ , i.e.

$$\Gamma\left(\frac{m}{2} - j - \frac{\alpha}{2}\right) = (-1)^j \frac{\pi}{\sin \pi\left(\frac{m}{2} - \frac{\alpha}{2}\right)} \frac{1}{\Gamma\left(-\frac{m}{2} + 1 + j + \frac{\alpha}{2}\right)},$$

it follows that

$$U_j(t, x, y) = \left( (-1)^j \frac{\pi}{\sin \pi\left(\frac{m}{2} - \frac{\alpha}{2}\right)} \frac{1}{\Gamma\left(-\frac{m}{2} + 1 + j + \frac{\alpha}{2}\right)} \right) \mathcal{U}_j(t, x, y),$$

so that the formula in odd spacetime dimensions becomes

$$\begin{aligned} & C_n \frac{1}{\sin \pi\left(\frac{m}{2} - \frac{\alpha}{2}\right)} \sum_{j=0}^{\infty} (-1)^j \frac{\mathcal{U}_j(t, x, y)}{\Gamma\left(-\frac{m}{2} + 1 + j + \frac{\alpha}{2}\right)} (t^2 - r^2)^{-\frac{m-2}{2}+j} \\ &= \int_0^{\infty} e^{i\theta(t^2-r^2)} \left( \sum_{j=0}^{\infty} \mathcal{U}_j(t, x, y) \theta_+^{\frac{n-3}{2}-j} \right) d\theta. \end{aligned} \tag{5.18}$$

The amplitude in the right side of (5.18) is then a formal analytic symbol,

$$A(t, x, y, \theta) = \sum_{j=0}^{\infty} \mathcal{U}_j(t, x, y) \theta_+^{\frac{n-3}{2}-j}, \tag{5.19}$$

Due to the Gamma-factors appearing in the identity (5.16), convergence of the series on the left side of (5.18) does not imply convergence of the series (5.19). However, there exists a realization of the formal symbol (5.19) by a holomorphic symbol

$$\mathcal{A}(t, x, y, \theta) = \sum_{0 \leq j \leq \frac{\theta}{\varepsilon C}} \mathcal{U}_j(t, x, y) \theta_+^{\frac{n-3}{2}-j},$$

and one obtains an analytic parametrix

$$U(t, x, y) = \int_0^{\infty} e^{i\theta\Gamma} \mathcal{A}(t, x, y, \theta) d\theta \tag{5.20}$$

which approximates the wave kernel for small  $|t|$  and  $(x, y)$  near the diagonal up to a holomorphic error, whose amplitude is exponentially decaying in  $\theta$ . Here, we recall (see [Sj], p. 3 and section 9) that a *classical formal analytic symbol* ([Sj], page 3) on a domain  $\Omega \subset \mathbb{C}^n$  is a formal semi-classical series

$$a(z, \lambda) = \sum_{k=0}^{\infty} a_k(z) \lambda^{-k},$$

where  $a_k(z, \lambda) \in \mathcal{O}(\Omega)$  for all  $\lambda > 0$ . Then for some  $C > 0$ , the  $a_k(z) \in \mathcal{O}(\Omega)$  satisfy

$$|a_k(z)| \leq C^{k+1} k^k, \quad k = 0, 1, 2, \dots$$

A realization of the formal symbol is a genuine holomorphic symbol of the form,

$$a(z, \lambda) = \sum_{0 \leq k \leq \frac{\lambda}{\varepsilon C}} a_k(z) \lambda^{-k}.$$

It is an analytic symbol since, with the index restriction,

$$|a_k(z)\lambda^{-k}| \leq C_\Omega \left(\frac{Ck}{\lambda}\right)^k \leq Ce^{-k}.$$

Hence the series converges uniformly on  $\Omega$  to a holomorphic function of  $z$  for each  $\lambda$ .

Returning to (5.19), the Hadamard-Riesz coefficients  $\mathcal{U}_j$  are determined inductively by the transport equations

$$(5.21) \quad \begin{cases} \frac{\Theta'}{2\Theta}\mathcal{U}_0 + \frac{\partial\mathcal{U}_0}{\partial r} = 0 \\ 4ir(x, y)\left\{\left(\frac{k+1}{r(x, y)} + \frac{\Theta'}{2\Theta}\right)\mathcal{U}_{j+1} + \frac{\partial\mathcal{U}_{j+1}}{\partial r}\right\} = \Delta_y\mathcal{U}_j. \end{cases},$$

whose solutions are given by:

$$(5.22) \quad \begin{cases} \mathcal{U}_0(x, y) = \Theta^{-\frac{1}{2}}(x, y) \\ \mathcal{U}_{j+1}(x, y) = \Theta^{-\frac{1}{2}}(x, y) \int_0^1 s^k \Theta(x, x_s)^{\frac{1}{2}} \Delta_2 \mathcal{U}_j(x, x_s) ds \end{cases}$$

where  $x_s$  is the geodesic from  $x$  to  $y$  parametrized proportionately to arc-length and where  $\Delta_2$  operates in the second variable.

As discussed above, the representation (5.18) does not suffice when  $n$  is odd, since  $\Gamma(z)$  and  $\theta_+^z$  have poles at the negative integers. To rescue the representation when  $n$  is odd, we need to use the distributions  $\theta_+^{-n}$  with  $n = 1, 2, \dots$ , defined as follows (see [HoI]):

$$\theta_+^{-k}(\varphi) = \int_0^\infty (\log \theta) \varphi^{(k)}(\theta) dx / (k-1)! + \varphi^{(k-1)}(0) \left( \sum_{j=1}^k 1/j \right) / (k-1)!.$$

This family behaves in an unusual way under derivation,

$$\frac{d}{d\theta} \theta_+^{-k} = -k\theta_+^{-k-1} + (-1)^k \delta_0^{(k)} / k!$$

(see [HoI](3.2.2)) and is therefore sometimes avoided in the Hadamard-Riesz parametrix construction (as in [Be]).

However, we have already constructed the parametrices and only want to express them in terms of the above oscillatory integrals to make contact with Fourier integral operator theory. In odd space dimensions, the Hadamard parametrices can be written in the form

$$(5.23) \quad \begin{aligned} & \int_0^\infty e^{i\theta\Gamma} (U_0(t, x, y)\theta_+^m + \dots + U_m\theta_+^0) d\theta \\ & + \int_0^\infty e^{i\theta\Gamma} (U_{m+1}\theta_+^{-1} + U_{m+2}\theta_+^{-2} + \dots) d\theta \end{aligned}$$

Again the amplitude is a formal symbol. To produce a genuine amplitude it needs to be replaced by a realization which approximates it modulo a holomorphic symbol which is exponentially decaying in  $\theta$ .

We are paying close attention to the regularization of the integral at  $\theta = 0$ , but only the behavior of the amplitude as  $\theta \rightarrow \infty$  is relevant to the singularity. The terms with  $\theta_+^{-k}$  for  $k > 0$  produce logarithmic terms in the kernel. If we use a smooth cutoff at  $\theta = 0$ , we obtain distributions of the form

$$u_\mu(\Gamma) = \int_{\mathbb{R}} e^{i\theta\Gamma} \chi(\theta) \theta^\mu d\theta$$

where  $\chi(\theta) = 1$  for  $\theta \geq 1$  and  $\chi(\theta) = 0$  for  $\theta \leq \frac{1}{2}$ . Then

$$u_{-k}(\Gamma) = i^{k+1}\Gamma^{k-1} \log \Gamma, \quad \text{modulo } C^\infty, \quad u_{-k}(-\Gamma) = (-i)^{n+1}\Gamma^{n-1} \log \Gamma.$$

Hence the terms with negative powers of  $\theta_+$  in (5.23) produce the logarithmic terms and the holomorphic terms.

Above, we discussed the Hadamard-Feynman fundamental solution, but for  $t > 0$  we only need to differentiate it in  $t$  (according to Proposition 5.1) to obtain the parametrices for the Poisson-wave group. Away from the characteristic conoid the Schwartz kernels of the Poisson-wave group and Hadamard-Feynman fundamental solution are holomorphic by the theorem on propagation of analytic wave front sets [Sj]. The Fourier integral structure and mapping properties follow immediately from the order of the amplitude and from the exact formula for the phase.

**9. Tempered spectral projector and Poisson semi-group as complex Fourier integral operators.** To study the tempered spectral projection kernels (4.12), we further need to continue  $U_{\mathbb{C}}(t, \zeta, y)$  anti-holomorphically in the  $y$  variable. The discussion is similar to the holomorphic case except that we need to double the Grauert tube radius to obtain convergence. We thus have (cf. (4.14))

$$\begin{aligned} U_{\mathbb{C}}(t + 2i\tau, \zeta, \bar{\zeta}) &= \sum_j e^{(-2\tau + it)\lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2 \\ &= \int_{\mathbb{R}} e^{it\lambda} d_{\lambda} P_{[0, \lambda]}^{\tau}(\zeta, \bar{\zeta}). \end{aligned} \tag{5.24}$$

Properties of these kernels may be obtained from kernels which are analytically continued in one variable only from the formula (4.15)

$$\begin{aligned} U_{\mathbb{C}}(t + 2i\tau, \zeta, \bar{\zeta}') &= \int_M U(t + i\tau, \zeta, y) U_{\mathbb{C}}(i\tau, y, \bar{\zeta}') dV_g(x) \\ &= \sum_j e^{(-2\tau + it)\lambda_j} \varphi_j^{\mathbb{C}}(\zeta) \overline{\varphi_j^{\mathbb{C}}(\zeta')}. \end{aligned} \tag{5.25}$$

We have,

**Proposition 5.2.** *For small  $t, \tau > 0$  and for sufficiently small  $\tau \geq \sqrt{\bar{\rho}(\zeta)} > 0$ , there exists a realization  $\mathcal{B}(t, \zeta, \bar{\zeta}, \theta)$  of a formal analytic symbol  $B(t, \zeta, \bar{\zeta}, \theta)$  so that as tempered distributions on  $\mathbb{R} \times M_{\tau}$ ,*

$$U_{\mathbb{C}}(t + 2i\tau, \zeta, \bar{\zeta}) = \int_0^{\infty} e^{i\theta((t+2i\tau) - 2i\sqrt{\bar{\rho}(\zeta)})} \mathcal{B}(t, \zeta, \bar{\zeta}, \theta) d\theta + R(t + 2i\tau, \zeta, \bar{\zeta}), \tag{5.26}$$

where  $R(t + 2i\tau, \zeta, \bar{\zeta})$  is the restriction to the anti-diagonal of a holomorphic kernel. Moreover

- $\theta((t + 2i\tau) - 2i\sqrt{\bar{\rho}(\zeta)})$  is a phase of positive type.
- If  $\sqrt{\bar{\rho}(\zeta)} < \tau$  the entire kernel is locally holomorphic.
- If  $\sqrt{\bar{\rho}(\zeta)} = \tau$  then

$$U_{\mathbb{C}}(t + 2i\tau, \zeta, \bar{\zeta}) = \int_0^{\infty} e^{i\theta t} \mathcal{B}(t, \zeta, \bar{\zeta}, \theta) d\theta + R(t + 2i\tau, \zeta, \bar{\zeta}). \tag{5.27}$$

*Proof.* We use the Hadamard parametrix (Corollary 5.1) for  $U(t + 2i\tau, \zeta, \bar{\zeta})$  and use (2.1) to simplify the phase, i.e. we write

$$\Gamma(t + 2i\tau, \zeta, \bar{\zeta}) = (t + 2i\tau - 2i\sqrt{\bar{\rho}})(t + 2i\tau + 2i\sqrt{\bar{\rho}})$$

in the Hadamard parametrix in Corollary 5.1. The factors of  $(t + 2i\tau + 2i\sqrt{\rho})$  are non-zero when  $\tau > 0$  and can be absorbed into the Hadamard coefficients. We denote the new amplitude by  $\mathcal{B}$  to distinguish it from the amplitude in Corollary 5.1. We then express each term as a Fourier integral distribution of complex type with phase  $t + 2i\tau - 2i\sqrt{\rho}$ . It is manifestly of positive type. On  $\partial M_\tau$ ,  $t + 2i\tau - 2i\sqrt{\rho}$  simplifies to  $t$ . □

**10. Complexified wave group and Szegő kernels.** As in [Ze07] it will also be necessary for us to understand the composition  $U_{\mathbb{C}}(i\tau)^*U_{\mathbb{C}}(i\tau)$ . In this regard, it is useful to introduce the Szegő kernels  $\Pi_\tau$  of  $M_\tau$ , i.e. the orthogonal projections

$$(5.28) \quad \Pi_\tau : L^2(\partial M_\tau, d\mu_\tau) \rightarrow H^2(\partial M_\tau, d\mu_\tau),$$

where  $d\mu_\tau$  is the Liouville volume form. Here as above,  $H^2(\partial M_\tau, d\mu_\tau)$  is the Hardy space of boundary values of holomorphic functions in  $M_\tau$  which belong to  $L^2(\partial M_\tau, d\mu_\tau)$ . It is simple to prove that the restrictions of  $\{\varphi_{\lambda_j}^{\mathbb{C}}\}$  to  $\partial M_\tau$  is a basis of  $H^2(\partial M_\tau, d\mu_\tau)$ . The Szegő projector  $\Pi_\tau$  is a complex Fourier integral operator with a positive complex canonical relation. The real points of its canonical relation form the graph  $\Delta_\Sigma$  of the identity map on the symplectic cone  $\Sigma_\tau \subset T^*\partial M_\tau$  (4.17). We refer to [Ze12] for further background. We only need the first statement in the following:

**Lemma 5.3.** *Let  $\Psi^s(X)$  denote the class of pseudo-differential operators of order  $s$  on  $X$ . Then,*

- $U_{\mathbb{C}}(i\tau)^*U_{\mathbb{C}}(i\tau) \in \Psi^{-\frac{m-1}{2}}(M)$  with principal symbol  $|\xi|_g^{-(\frac{m-1}{2})}$ .
- $U_{\mathbb{C}}(i\tau) \circ U_{\mathbb{C}}(i\tau)^* = \Pi_\tau A_\tau \Pi_\tau$  where  $A_\tau \in \Psi^{\frac{m-1}{2}}(\partial M_\tau)$  has principal symbol  $|\sigma|_g^{(\frac{m-1}{2})}$  as a function on  $\Sigma_\tau$ .

*Proof.* This follows from Proposition 4.2. The first statement is a special case of the following Lemma from [Ze07]: Let  $a \in S^0(T^*M - 0)$ . Then for all  $0 < \tau < \tau_{\max}(g)$ , we have:

$$U(i\tau)^* \Pi_\tau a \Pi_\tau U(i\tau) \in \Psi^{-\frac{m-1}{2}}(M),$$

with principal symbol equal to  $a(x, \xi) |\xi|_g^{-(\frac{m-1}{2})}$

The second statement follows from Theorem 4.2 and the composition theorem for complex Fourier integral operators. We note that

$$(5.29) \quad U_{\mathbb{C}}(i\tau) \circ U_{\mathbb{C}}(i\tau)^*(\zeta, \zeta') = \sum_j e^{-2\tau\lambda_j} \varphi_{\lambda_j}^{\mathbb{C}}(\zeta) \overline{\varphi_{\lambda_j}^{\mathbb{C}}(\zeta')}.$$

□

## 6. GROWTH OF COMPLEXIFIED EIGENFUNCTIONS

**1. A Siciak-Zaharjuta extremal function for Grauert tubes.** Before defining the analogues, let us first recall the definitions of relative maximal or extremal PSH functions satisfying bounds on a pair  $E \subset \Omega \subset \mathbb{C}^m$  where  $\Omega$  is a bounded open set. There are two definitions:

- The pluri-complex Green's function relative to a subset  $E \subset \Omega$ , defined <sup>[Sic]</sup> as the upper semi-continuous regularization  $V_{E,\Omega}^*$  of

$$V_{E,\Omega}(z) = \sup\{u(z) : u \in PSH(\Omega), u|_E \leq 0, u|_{\partial\Omega} \leq 1\} .$$

- The Siciak-Zaharjuta extremal function relative to  $E \subset \Omega$ , defined by

$$\log \Phi_E^N(\zeta) = \sup\left\{\frac{1}{N} \log |p_N(\zeta)| : p \in \mathcal{P}_E^N\right\}, \quad \log \Phi_E = \limsup_{N \rightarrow \infty} \log \Phi_E^N,$$

$$\text{where } \mathcal{P}_E^N = \{p \in \mathcal{P}^N : \|p\|_E \leq 1, \|p\|_\Omega \leq e^N\}.$$

Here,  $\|f\|_E = \sup_{z \in E} |f(z)|$  and  $\mathcal{P}^N$  denotes the space of all complex analytic polynomials of degree  $N$ . Siciak proved that  $\log \Phi_E = V_E$  (see <sup>[Sic]</sup> Theorem 1). Intuitively, there are enough polynomials that one can obtain the sup by restricting to polynomials.

There are analogous definitions in the case of unit co-disc bundles in the dual of a positive holomorphic Hermitian line bundle  $L \rightarrow M$  over a Kähler manifold. In the case of  $\mathbb{C}\mathbb{P}^n$ , one defines

$$V_K(z) = \sup\{u(z) : u \in \mathcal{L}, u \leq 0 \text{ on } K\}$$

where  $\mathcal{L}$  denotes the Lelong class of all global plurisubharmonic (PSH) functions  $u$  on  $\mathbb{C}^n$  with  $u(z) \leq c_u + \log(1 + |z|)$ .

We now define an analogue of the Siciak-Zaharjuta extremal function for Grauert tubes in the special case where  $E = M$ , the underlying real manifold. The Riemannian analogue of  $\mathcal{P}^N$  is the space

$$\mathcal{H}^\lambda = \left\{p = \sum_{j:\lambda_j \in I_\lambda} a_j \varphi_{\lambda_j}^{\mathbb{C}}, \quad a_1, \dots, a_{N(\lambda)} \in \mathbb{R}\right\}$$

spanned by the eigenfunctions with 'degree'  $\lambda_j \leq \lambda$ . Here,  $N(\lambda) = \#\{j : \lambda_j \in I_\lambda\}$ . As above, we could let  $I_\lambda = [0, \lambda]$  or  $I_\lambda = [\lambda, \lambda + c]$  for some  $c > 0$ . It is simpler to work with  $L^2$  based norms than sup norms, and so we define

$$S\mathcal{H}_M^\lambda = \left\{\psi = \sum_{j:\lambda_j \leq \lambda} a_j \varphi_{\lambda_j}^{\mathbb{C}}, \quad \sum_{j=1}^{N(\lambda)} |a_j|^2 = 1\right\}.$$

**Definition 6.1.** The Riemannian Siciak-Zaharjuta extremal function (with respect to the real locus  $M$ ) is defined by:

$$(6.1) \quad \begin{cases} \log \Phi_M^\lambda(\zeta) = \sup\left\{\frac{1}{\lambda} \log |\psi(\zeta)| : \psi \in S\mathcal{H}_M^\lambda\right\}, \\ \log \Phi_M = \limsup_{\lambda \rightarrow \infty} \log \Phi_M^\lambda. \end{cases}$$

**Remark 6.2.** One could define the analogous notion for any set  $E \subset M_\tau$ , with

$$S\mathcal{H}_E^\lambda = \{p \in \mathcal{H}^\lambda, \|p\|_{L^2(E)} \leq 1\}.$$

But we only discuss the results for  $E = M$ .

One could also define the pluri-complex Green's function of  $M_\tau$  as follows:

**Definition 6.3.** Let  $(M, g)$  be a real analytic Riemannian manifold, let  $M_\tau$  be an open Grauert tube, and let  $E \subset M_\tau$ . The Riemannian pluri-complex Green's function with respect to  $(E, M_\tau, g)$  is defined by

$$V_{g,E,\tau}(\zeta) = \sup\{u(z) : u \in PSH(M_\tau), u|_E \leq 0, u|_{\partial M_\tau} \leq \tau\}.$$

It is obvious that  $V_{g,E,\tau}(\zeta) \geq \sqrt{\rho}(\zeta)$  and it is almost standard that  $V_{g,M,\tau}(\zeta) = \sqrt{\rho}(\zeta)$ . See Proposition 4.1 of [GZ] or Corollary 9 of [BT82]. The set  $M = (\sqrt{\rho})^{-1}(0)$  is often called the center. As proved in [LS1], there are no smooth exhaustion functions solving the exact HCMA (Theorem 1.1). Hence  $u$  must be singular on its minimum set. In [HW] it is proved that the minimum set of strictly PSH function is totally real.

**2. Statement of results.** Our first results concern the logarithmic asymptotics of the complexified spectral projections.

**SICIAK** **Theorem 6.4.** (see also [Ze12]) Let  $I_\lambda = [0, \lambda]$ . Then

- (1)  $\log \Phi_M^\lambda(\zeta) = \frac{1}{\lambda} \log \Pi_{I_\lambda}^{\mathbb{C}}(\zeta, \bar{\zeta})$ .
- (2)  $\log \Phi_M = \lim_{\lambda \rightarrow \infty} \log \Phi_M^\lambda = \sqrt{\rho}$ .

To prove the Theorem, it is convenient to study the tempered spectral projection measures (4.13), or in differentiated form (4.9),

$$(6.2) \quad d_\lambda P_{[0,\lambda]}^\tau(\zeta, \bar{\zeta}) = \sum_j \delta(\lambda - \lambda_j) e^{-2\tau\lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2,$$

which is a temperate distribution on  $\mathbb{R}$  for each  $\zeta$  satisfying  $\sqrt{\rho}(\zeta) \leq \tau$ . When we set  $\tau = \sqrt{\rho}(\zeta)$  we omit the  $\tau$  and write (as in (4.10)),

$$(6.3) \quad d_\lambda P_{[0,\lambda]}(\zeta, \bar{\zeta}) = \sum_j \delta(\lambda - \lambda_j) e^{-2\sqrt{\rho}(\zeta)\lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2.$$

The advantage of the tempered projections is that they have polynomial asymptotics and one can use standard Tauberian theorems to analyse their growth.

We prove the following one-term local Weyl law for complexified spectral projections:

**PTAULWL** **Theorem 6.5.** On any compact real analytic Riemannian manifold  $(M, g)$  of dimension  $n$ , we have, with remainders uniform in  $\zeta$ ,

- (1) For  $\sqrt{\rho}(\zeta) \geq \frac{C}{\lambda}$ ,

$$P_{[0,\lambda]}(\zeta, \bar{\zeta}) = (2\pi)^{-n} \left( \frac{\lambda}{\sqrt{\rho}} \right)^{\frac{n-1}{2}} \left( \frac{\lambda}{(n-1)/2 + 1} + O(1) \right);$$

- (2) For  $\sqrt{\rho}(\zeta) \leq \frac{C}{\lambda}$ ,

$$P_{[0,\lambda]}(\zeta, \bar{\zeta}) = (2\pi)^{-n} \lambda^n (1 + O(\lambda^{-1})).$$

This implies new bounds on pointwise norms on complexified eigenfunctions, improving those of [GLS]. inequality gives

**PWa** **Corollary 6.6.** Suppose  $(M, g)$  is real analytic of dimension  $n$ , and that  $I_\lambda = [0, \lambda]$ . Then,

(1) For  $\tau \geq \frac{C}{\lambda}$  and  $\sqrt{\rho}(\zeta) = \tau$ , there exists  $C > 0$  so that

$$C\lambda_j^{-\frac{n-1}{2}} e^{\tau\lambda} \leq \sup_{\zeta \in M_\tau} |\varphi_\lambda^{\mathbb{C}}(\zeta)| \leq C\lambda^{\frac{n-1}{4} + \frac{1}{2}} e^{\tau\lambda}.$$

(2) For  $\tau \leq \frac{C}{\lambda}$ , and  $\sqrt{\rho}(\zeta) = \tau$ , there exists  $C > 0$  so that

$$|\varphi_\lambda^{\mathbb{C}}(\zeta)| \leq \lambda^{\frac{n-1}{2}};$$

The lower bound of Corollary [6.6](#) (1) combines Theorem [6.5](#) with Gårding's inequality. The upper bound sharpens the estimates claimed in [\[Bou, GLS\]](#),

$$(6.4) \quad \sup_{\zeta \in M_\tau} |\varphi_\lambda^{\mathbb{C}}(\zeta)| \leq C_\tau \lambda^{n+1} e^{\tau\lambda}.$$

The improvement is due to using spectral asymptotics rather than a crude Sobolev inequality.

**3. Siciak extremal functions: Proof of Theorem [6.4](#) (1).** In this section we prove Theorem [6.4](#). First we prove a pointwise local Weyl law in the complex domain.

**4. Proof of Theorem [6.4](#) (2).** This follows from Theorem [6.5](#) together with the following

[\[Ze12\]](#) For any  $\tau = \sqrt{\rho}(\zeta) > 0$ , and for any  $\delta > 0$ ,

$$2\sqrt{\rho}(\zeta) - \frac{\log |\delta|}{\lambda} + O\left(\frac{\log \lambda}{\lambda}\right) \leq \frac{1}{\lambda} \log \Pi_{[0,\lambda]}(\zeta, \bar{\zeta}) \leq 2\sqrt{\rho}(\zeta) + O\left(\frac{\log \lambda}{\lambda}\right)$$

hence

$$\lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \log \Pi_{[0,\lambda]}(\zeta, \bar{\zeta}) = 2\sqrt{\rho}(\zeta).$$

**Lemma 6.7.** *Proof.* For the upper bound, we use that

$$(6.5) \quad \begin{aligned} \Pi_{[0,\lambda]}(\zeta, \bar{\zeta}) &\leq e^{2\lambda\sqrt{\rho(\zeta)}} \sum_{j:\lambda_j \in [0,\lambda]} e^{-2\sqrt{\rho(\zeta)}\lambda_j} |\varphi_{\lambda_j}^{\mathbb{C}}(\zeta)|^2 \\ &= e^{2\lambda\sqrt{\rho(\zeta)}} P_{[0,\lambda]}(\zeta, \bar{\zeta}). \end{aligned}$$

We then take  $\frac{1}{\lambda}$  log of both sides and apply Theorem [6.5](#) to conclude the proof.

The lower bound is subtler for reasons having to do with the distribution of eigenvalues (see the Remark below). It is most natural to prove two-term Weyl asymptotics for  $P_{[0,\lambda]}(\zeta, \bar{\zeta})$  and to deduce Weyl asymptotics for short spectral intervals  $[\lambda, \lambda + 1]$ . But that requires an analysis of the singularity of the trace of the complexified wave group for longer times than a short interval around  $t = 0$  and we postpone the more refined analysis until [\[Zew\]](#).

Instead we use the longer intervals  $[(1 - \delta)\lambda, \lambda]$  for some  $\delta > 0$ . We clearly have

$$(6.6) \quad e^{2(1-\delta)\lambda\sqrt{\rho(\zeta)}} \sum_{j:\lambda_j \in [(1-\delta)\lambda, \lambda]} e^{-2\sqrt{\rho(\zeta)}\lambda_j} |\varphi_{\lambda_j}^{\mathbb{C}}(\zeta)|^2 \leq \Pi_{[0,\lambda]}(\zeta, \bar{\zeta})$$

By Theorem [6.5](#),

$$\begin{aligned} \sum_{j:\lambda_j \in [(1-\delta)\lambda, \lambda]} e^{-2\sqrt{\rho(\zeta)}\lambda_j} |\varphi_{\lambda_j}^{\mathbb{C}}(\zeta)|^2 &= P_{[0,\lambda]}(\zeta, \bar{\zeta}) - P_{[0,(1-\delta)\lambda]}(\zeta, \bar{\zeta}) \\ &= C_n(\tau)[1 - (1 - \delta)^n] \lambda^{\frac{n+1}{2}} + O(\lambda^{\frac{n-1}{2}}) \end{aligned}$$

Taking  $\frac{1}{\lambda} \log$  then gives

$$\frac{1}{\lambda} \log \Pi_{[0,\lambda]}(\zeta, \bar{\zeta}) \geq 2(1 - \delta) \sqrt{\rho}(\zeta) - \frac{|\log \delta|}{\lambda} + O\left(\frac{\log \lambda}{\lambda}\right).$$

It follows that for all  $\delta > 0$ ,

$$\liminf_{\lambda \rightarrow \infty} \frac{1}{\lambda} \log \Pi_{[0,\lambda]}(\zeta, \bar{\zeta}) \geq 2(1 - \delta) \sqrt{\rho}(\zeta).$$

The conclusion of the Lemma follows from the fact that the left side is independent of  $\delta$ .  $\square$

**Remark 6.8.** *The problematic issue in the lower bound is the width of  $I_\lambda$ . If  $(M, g)$  is a Zoll manifold, the eigenvalues of  $\sqrt{\Delta}$  form clusters of width  $O(\lambda^{-1})$  around an arithmetic progression  $\{k + \frac{\beta}{4}\}$  for a certain Morse index  $\beta$ . Unless the intervals  $I_\lambda$  are carefully centered around this progression,  $P_{I_\lambda}$  could be zero. Hence we must use long spectral intervals if we do not analyze the long time behavior of the geodesic flow; for short ones no general lower bound exists.*

## 5. Proof of Theorem <sup>SICIAK</sup>6.4 (1).

*Proof.* We need to show that

$$\Pi_{I_\lambda}^{\mathbb{C}}(\zeta, \bar{\zeta}) = \sup\{|\varphi(\zeta)|^2 : \varphi = \sum_{j:\lambda_j \in I} a_j \varphi_{\lambda_j}^{\mathbb{C}}, \quad \|a\| = 1\}.$$

We define the ‘coherent state’,

$$\Phi_\lambda^z(w) = \frac{\Pi_{I_\lambda}^{\mathbb{C}}(w, \bar{z})}{\sqrt{\Pi_{I_\lambda}^{\mathbb{C}}(z, \bar{z})}},$$

satisfying,

$$\Phi_\lambda^z(w) = \sum_{j:I_\lambda} a_j \varphi_j^{\mathbb{C}}(w), \quad a_j = \frac{\overline{\varphi_j^{\mathbb{C}}(\zeta)}}{\sqrt{\Pi_{I_\lambda}^{\mathbb{C}}(z, \bar{z})}}, \quad \sum_j |a_j|^2 = 1.$$

Hence,  $\Phi_{I_\lambda}^\zeta$  is a competitor for the sup and since  $|\Phi_{I_\lambda}^\zeta(\zeta)|^2 = \Pi_{I_\lambda}(\zeta, \bar{\zeta})$  one has

$$\Pi_{I_\lambda}^{\mathbb{C}}(\zeta, \bar{\zeta}) \leq \sup\{|\psi(\zeta)|^2 : \psi = \sum_{j:\lambda_j \in I} a_j \varphi_j^{\mathbb{C}}, \quad \|a\| = 1\}.$$

On the other hand, by the Schwartz inequality for  $\ell^2$ , for any  $\psi = \sum_{j:\lambda_j \in I} a_j \varphi_j^{\mathbb{C}}$  one has

$$\left| \sum_{j:\lambda_j \in I} a_j \varphi_j^{\mathbb{C}} \right|^2 = |\langle a, \psi \rangle|^2 \leq \|a\|^2 \sum |\varphi_j^{\mathbb{C}}|^2 = \Pi_{I_\lambda}(\zeta, \bar{\zeta})$$

and one has

$$\Pi_I^{\mathbb{C}}(\zeta, \bar{\zeta}) \geq \sup\{|\psi(\zeta)|^2 : \psi = \sum_{j:\lambda_j \in I} a_j \varphi_j^{\mathbb{C}}, \quad \|a\| = 1\}.$$

$\square$

**Remark 6.9.** Since  $N(I_\lambda) \sim \lambda^{m-1}$ ,

$$\begin{aligned} \frac{1}{\lambda} \log \Pi_{I_\lambda}(\zeta, \bar{\zeta}) &= \frac{1}{\lambda} \log \left( \sum_{j: \lambda_j \in I_\lambda} |\varphi_{\lambda_j}^{\mathbb{C}}(\zeta)|^2 \right) \\ &= \max_{j: \lambda_j \in I_\lambda} \left\{ \frac{1}{\lambda} \log |\varphi_{\lambda_j}^{\mathbb{C}}(\zeta)|^2 \right\} + O\left(\frac{\log \lambda}{\lambda}\right). \end{aligned}$$

We recall that a sequence of eigenfunctions is called quantum ergodic if  $\langle A\varphi_j, \varphi_j \rangle \rightarrow \frac{1}{\mu(S_g^*M)} \int_{S_g^*M} \sigma_A d\mu$ . The complexified eigenfunctions then satisfy  $\frac{1}{\lambda_j} \log |\varphi_j(\zeta)| \rightarrow \sqrt{\rho}(\zeta)$ . It follows that ergodic eigenfunctions are asymptotically maximal, i.e. have the same logarithmic asymptotics as  $\Phi_M^\lambda$ .

## 7. POINTWISE PHASE SPACE WEYL LAWS ON GRAUERT TUBES

**1. Two term pointwise Weyl laws in Grauert tubes.** The asymptotics of the complexified spectral projection kernels (4.13) are complex analogues of those of the diagonal spectral projections in the real domain and reflect the structure of complex geodesics from  $\zeta$  to  $\bar{\zeta}$ . As in the real domain, one can obtain more refined asymptotics of  $P_{[\lambda, \lambda+1]}(\zeta, \bar{\zeta})$  by using the structure of geodesic segments from  $\zeta$  to  $\bar{\zeta}$ . This is the subject of the work in progress [ZeW]. For the sake of completeness, we state the results here: There exists an explicit complex oscillatory factor  $Q_\zeta(\lambda)$  depending on the geodesic arc from  $\zeta$  to  $\bar{\zeta}$  such that

(1) For  $\sqrt{\rho}(\zeta) \geq \frac{C}{\lambda}$ ,

$$P_{[0, \lambda]}^\tau(\zeta, \bar{\zeta}) = (2\pi)^{-n} \lambda \left( \frac{\lambda}{\sqrt{\rho}} \right)^{\frac{n-1}{2}} (1 + Q_\zeta(\lambda) \lambda^{-1} + o(\lambda^{-1}));$$

(2) For  $\sqrt{\rho}(\zeta) \leq \frac{C}{\lambda}$ ,

$$P_{[0, \lambda]}^\tau(\zeta, \bar{\zeta}) = (2\pi)^{-n} \lambda^n + Q_\zeta(\lambda) \lambda^{n-1} + o(\lambda^{n-1}),$$

In this section, we prove Theorem 6.5 (I). To prove the local Weyl law we employ parametrices for the Poisson-wave kernel adapted to  $e^{i(t+i\tau)\sqrt{\Delta}}$  for  $\tau > 0$  which are best adapted to the complex geometry.

*Proof.* As in the real domain, we obtain asymptotics of  $P_{[0, \lambda]}^\tau(\zeta, \bar{\zeta})$  by the Fourier-Tauberian method of relating their asymptotics to the singularities in the real time  $t$  of the Fourier transform (4.14). We refer to [SV] for background on Tauberian theorems. We follow the classical argument of [DG], Proposition 2.1, to obtain the local Weyl law with remainder one degree below the main term.

The proof is based on the oscillatory integral representation of Proposition 5.2. We are working in the case where  $\sqrt{\rho}(\zeta) = \tau$  and hence can simplify it to (5.27).

We then introduce a cutoff function  $\psi \in \mathcal{S}(\mathbb{R})$  with  $\hat{\psi} \in C_0^\infty$  supported in sufficiently small neighborhood of 0 so that no other singularities of  $U_{\mathbb{C}}(t + 2i\tau, \zeta, \bar{\zeta})$  lie in its support. We also assume  $\hat{\psi} \equiv 1$  in a smaller neighborhood of 0. We then change variables  $\theta \rightarrow \lambda\theta$  and

apply the complex stationary phase to the integral,

CXPARAONEb

$$(7.1) \quad \begin{aligned} & \int_{\mathbb{R}} \hat{\psi}(t) e^{-i\lambda t} U_{\mathbb{C}}(t + 2i\tau, \zeta, \bar{\zeta}) dt \\ &= \int_{\mathbb{R}} \int_0^{\infty} \hat{\psi}(t) e^{-i\lambda t} e^{i\theta t} (\mathcal{B}(t, \zeta, \bar{\zeta}, \theta) d\theta + R(t + 2i\tau, \zeta, \bar{\zeta})) dt. \end{aligned}$$

The second  $R$  term can be dropped since it is of order  $\lambda^{-M}$  for all  $M > 0$ . In the first we change variables  $\theta \rightarrow \lambda\theta$  to obtain a semi-classical Fourier integral distribution of real type with phase  $e^{i\lambda t(\theta-1)}$ . The critical set consists of  $\theta = 1, t = 0$ . The phase is clearly non-degenerate with Hessian determinant one and inverse Hessian operator  $D_{\theta,t}^2$ . Taking into account the factor of  $\lambda^{-1}$  from the change of variables, the stationary phase expansion gives

PANSIONCa

$$(7.2) \quad \sum_j \psi(\lambda - \lambda_j) e^{-2\tau\lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2 \sim \sum_{k=0}^{\infty} \lambda^{\frac{n-1}{2}-k} \omega_k(\tau; \zeta)$$

where the coefficients  $\omega_k(\tau, \zeta)$  are smooth for  $\zeta \in \partial M_{\tau}$ . However the coefficients are not uniform as  $\tau \rightarrow 0^+$  due to the factors of  $(t+2i\tau+2i\sqrt{\rho}(\zeta))$  which were left in the denominators of the modified Hadamard parametrix. Since  $t = 0$  at the stationary phase point, the resulting expansion is equivalent to one with the large parameter  $\tau\lambda$  (or  $\sqrt{\rho}(\zeta)\lambda$ ). The uniform expansion is then

EXPANSIONCa

$$(7.3) \quad \sum_j \psi(\lambda - \lambda_j) e^{-2\tau\lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2 \sim \sum_{k=0}^{\infty} \left(\frac{\lambda}{\tau}\right)^{\frac{n-1}{2}-k} \omega_k(\zeta, \bar{\zeta}),$$

where  $\omega_j$  are smooth in  $\zeta$ , and  $\omega_0 = 1$ . The remainder has the same form.

To complete the proof, we apply the Fourier Tauberian theorem (see the Appendix ([SV]): Let  $N \in F_+$  and let  $\psi \in \mathcal{S}(\mathbb{R})$  satisfy the conditions:  $\psi$  is even,  $\psi(\lambda) > 0$  for all  $\lambda \in \mathbb{R}$ ,  $\hat{\psi} \in C_0^{\infty}$ , and  $\hat{\psi}(0) = 1$ . Then,

$$\psi * dN(\lambda) \leq A\lambda^{\nu} \implies |N(\lambda) - N * \psi(\lambda)| \leq CA\lambda^{\nu},$$

where  $C$  is independent of  $A, \lambda$ . We apply it twice, first in the region  $\sqrt{\rho}(\zeta) \geq C\lambda^{-1}$  and second in the complementary region.

In the first region, we let  $N_{\tau,\zeta}(\lambda) = P_{\tau,\lambda}(\zeta, \bar{\zeta})$ . It is clear that for  $\sqrt{\rho} = \tau$ ,  $N_{\tau,\zeta}(\lambda)$  is a monotone non-decreasing function of  $\lambda$  of polynomial growth which vanishes for  $\lambda \leq 0$ . For  $\psi \in \mathcal{S}$  positive, even and with  $\hat{\psi} \in C_0^{\infty}(\mathbb{R})$  and  $\hat{\psi}(0) = 1$ , we have by ([7.3]) that

PSIEST

$$(7.4) \quad \psi * dN_{\tau,\zeta}(\lambda) \leq C \left(\frac{\lambda}{\tau}\right)^{\frac{n-1}{2}},$$

where  $C$  is independent of  $\zeta, \lambda$ . It follows by the Fourier Tauberian theorem that

$$N_{\tau,\zeta}(\lambda) = N_{\tau,\zeta}(\lambda) * \psi(\lambda) + O\left(\frac{\lambda}{\tau}\right)^{\frac{n-1}{2}}.$$

Further, by integrating ([7.3]) from 0 to  $\lambda$  we have

$$N_{\tau,\zeta}(\lambda) * \psi(\lambda) = \left(\frac{\lambda}{\tau}\right)^{\frac{n-1}{2}} \left(\frac{\lambda}{\frac{n-1}{2} + 1} + O(1)\right),$$

proving (1).

To obtain uniform asymptotics in  $\tau$  down to  $\tau = 0$ , we use instead the analytic continuation of the Hörmander parametrrix (4.6). We choose local coordinates near  $x$  and write  $\exp_x^{-1}(y) = \Psi(x, y)$  in these local coordinates for  $y$  near  $x$ , and write the integral  $T_y^*M$  as an integral over  $\mathbb{R}^m$  in these coordinates. The holomorphic extension of the parametrrix to the Grauert tube  $|\zeta| < \tau$  at time  $t + 2i\tau$  has the form (4.7)–(5.27), i.e.

$$(7.5) \quad U_{\mathbb{C}}(t + 2i\tau, \zeta, \bar{\zeta}) = \int_{\mathbb{R}^n} e^{(it-2\tau)|\xi|_{g_y}} e^{i\langle \xi, \Psi(\zeta, \bar{\zeta}) \rangle} A(t, \zeta, \bar{\zeta}, \xi) d\xi.$$

Again, we use a cutoff function  $\psi \in \mathcal{S}(\mathbb{R})$  with  $\hat{\psi} \in C_0^\infty$  supported in sufficiently small neighborhood of 0 so that no other singularities of  $E(t + 2i\tau, \zeta, \bar{\zeta})$  lie in its support and so that  $\hat{\psi} \equiv 1$  in a smaller neighborhood of 0. We write the integral in polar coordinates and obtain

$$(7.6) \quad \begin{aligned} & \int_{\mathbb{R}} \hat{\psi}(t) e^{-i\lambda t} U_{\mathbb{C}}(t + 2i\tau, \zeta, \bar{\zeta}) dt \\ &= \lambda^m \int_0^\infty \int_{\mathbb{R}} \hat{\psi}(t) e^{-i\lambda t} \int_{S^{n-1}} e^{(it-2\tau)\lambda r} e^{ir\lambda \langle \omega, \Psi(\zeta, \bar{\zeta}) \rangle} A(t, \zeta, \bar{\zeta}, \lambda r \omega) r^{n-1} dr d\omega. \end{aligned}$$

We then apply complex stationary phase to the  $dr dt$  integral, regarding

$$\int_{S^{n-1}} e^{ir\lambda \langle \omega, \Psi(\zeta, \bar{\zeta}) \rangle} A(t, \zeta, \bar{\zeta}, \lambda r \omega) r^{m-1} d\omega$$

as the amplitude. When  $\sqrt{\rho}(\zeta) \leq \frac{C}{\lambda}$  the exponent is bounded in  $\lambda$  and the integral defines a symbol. Applying stationary phase again to the  $dt d\theta$  integral now gives

$$(7.7) \quad \sum_j \psi(\lambda - \lambda_j) e^{-2\tau\lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2 \sim \sum_{k=0}^\infty \lambda^{n-1-k} \omega_k(\zeta, \bar{\zeta}),$$

where  $\omega_k(\zeta, \bar{\zeta})$  is smooth down to the zero section.

We apply the Fourier Tauberian theorem again, but this time with the estimates

$$\psi * dN_{\tau, \zeta}(\lambda) \leq C\lambda^{n-1},$$

where  $C$  is independent of  $\zeta$ . We conclude that

$$N_{\tau, \zeta}(\lambda) = C\lambda^n + O(\lambda^{n-1}),$$

proving (2). □

**EASY** **Corollary 7.1.** *For all  $\zeta \in M_{\mathbb{C}}$ , and with  $\tau = \sqrt{\rho}(\zeta)$ ,*

$$c\lambda^{\frac{n+1}{2}} \leq P_{[0, \lambda]}^\tau(\zeta, \bar{\zeta}) \leq C\lambda^n.$$

## 2. Proof of Corollary <sup>PWa</sup> 6.6.

*Proof.* For the upper bound, we use that

$$\sup_{\zeta \in \partial M_\tau} |\varphi_\lambda^{\mathbb{C}}(\zeta)|^2 \leq \sup_{\zeta \in \partial M_\tau} |\Pi_{I_\lambda}(\zeta, \bar{\zeta})| \leq \sup_{\zeta \in \partial M_\tau} e^{\lambda\sqrt{\rho}(\zeta)} |P_{I_\lambda}(\zeta)|.$$

The upper bound stated in Corollary <sup>PWa</sup> 6.6 then follows from Corollary <sup>EASY</sup> 7.1 to Theorem <sup>PTAULWL</sup> 6.5.

For the lower bound in (2) of Corollary <sup>PWa</sup>6.6, we use that

$$\|\varphi_j^{\mathbb{C}}\|_{L^2(\partial M_\tau)} = e^{2\tau_j} \langle U(i\tau)^* U(i\tau) \varphi_j, \varphi_j \rangle_{L^2(M)}.$$

By Lemma <sup>PSIDostuff</sup>5.3, the operator  $U(i\tau)^* U(i\tau)$  is an elliptic pseudodifferential operator of order  $\mu = -\frac{n-1}{2}$  (or so). Let  $C > 0$  be a lower bound for its symbol times  $\langle \xi \rangle^\mu$ . Then by Garding's inequality,

$$\langle U(i\tau)^* U(i\tau) \varphi_j, \varphi_j \rangle_{L^2(M)} \geq C \lambda_j^{-\mu},$$

and so

$$\boxed{\text{GARDING}} \quad (7.8) \quad \|\varphi_j^{\mathbb{C}}\|_{L^2(\partial M_\tau)} \geq C \lambda_j^{-\mu} e^{2\tau \lambda_j}.$$

□

XNODALSECT

## 8. COMPLEX NODAL SETS AND SEQUENCES OF LOGARITHMS

In Theorem <sup>ERGXYZ</sup>7.7, we regard the zero set  $[Z_f]$  as a *current of integration*, i.e. as a linear functional on  $(m-1, m-1)$  forms  $\psi$

$$\langle [Z_{\varphi_j}], \psi \rangle = \int_{Z_{\varphi_j}} \psi.$$

Recall that a *current* is a linear functional (distribution) on smooth forms. We refer to <sup>GH</sup>[[GH]] <sup>MEAS</sup>for background. On a complex manifold one has  $(p, q)$  forms with  $p$   $dz_j$  and  $q$   $d\bar{z}_k$ 's. In <sup>(??)</sup>we use the Kähler hypersurface volume form  $\omega_g^{m-1}$  (where  $\omega_g = i\partial\bar{\partial}\rho$ ) to make  $Z_{\varphi_j}$  into a measure:

$$\langle [Z_{\varphi_j}], f \rangle = \int_{Z_{\varphi_j}} f \omega_g^{m-1}, \quad (f \in C(M)).$$

PLLSEC

1. **Poincaré-Lelong formula.** One of the two key reasons for the gain in simplicity is that there exists a simple analytical formula for the delta-function on the nodal set. The *Poincaré-Lelong formula* gives an exact formula for the delta-function on the zero set of  $\varphi_j$

$$\boxed{\text{PLLb}} \quad (8.1) \quad \frac{i}{2\pi} \partial\bar{\partial} \log |\varphi_j^{\mathbb{C}}(z)|^2 = [\mathcal{N}_{\varphi_j^{\mathbb{C}}}].$$

Thus, if  $\psi$  is an  $(n-1, n-1)$  form,

$$\int_{\mathcal{N}_{\varphi_j^{\mathbb{C}}}} \psi = \frac{1}{2\pi} \int_{M_\varepsilon} \psi \wedge i\partial\bar{\partial} \log |\varphi_j^{\mathbb{C}}(z)|^2.$$

PSHSECT

2. **Sequences of pluri-subharmonic functions and a weak\* limit problem for  $\frac{1}{\lambda} \log |\varphi_j^{\mathbb{C}}|^2$ .** We next consider logarithms of Husimi functions, which are PSH = (pluri-subharmonic) functions on  $M_\varepsilon$ . A function  $f$  on a domain in a complex manifold is PSH if  $i\partial\bar{\partial}f$  is a *positive (1,1) current*. That is,  $i\partial\bar{\partial}f$  is a singular form of type  $\sum_{i\bar{j}} a_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$  with  $(a_{j\bar{k}})$  positive definite Hermitian. If  $f$  is a local holomorphic function, then  $\log |f(z)|$  is PSH and  $i\partial\bar{\partial} \log |f(z)| = [Z_f]$ . General references are <sup>GH, HoC</sup>[[GH, HoC]].

A sequence of  $(1, 1)$  currents  $E_k$  converges weak\* to a current  $E$  if  $\langle E_k, \psi \rangle \rightarrow \langle E, \psi \rangle$  for all smooth  $(m-1, m-1)$  forms. Thus, for all  $f$

$$[Z_{\varphi_j}] \rightarrow i\partial\bar{\partial}\sqrt{\rho} \iff \int_{Z_{\varphi_j}} f\omega^{m-1} \rightarrow i \int_{M_\varepsilon} f\partial\bar{\partial}\sqrt{\rho} \wedge \omega^{m-1, m-1}.$$

**3. Pluri-subharmonic functions and compactness.** In the real domain, we have emphasized the problem of finding quantum limits (or microlocal defect measures). The same problem exists in the complex domain for the sequence of Husimi functions (1.2). However, there also exists a new problem involving the sequence of normalized logarithms

**LOGS** (8.2) 
$$\{u_j := \frac{1}{\lambda_j} \log |\varphi_j^{\mathbb{C}}(z)|^2\}_{j=1}^\infty.$$

A key fact is that this sequence is pre-compact in  $L^p(M_\varepsilon)$  for all  $p < \infty$  and even that

(8.3) 
$$\{\frac{1}{\lambda_j} \nabla \log |\varphi_j^{\mathbb{C}}(z)|^2\}_{j=1}^\infty.$$

is pre-compact in  $L^1(M_\varepsilon)$ .

**HARTOGS**

**Lemma 8.1.** (Hartog's Lemma; (see <sup>HoI</sup>HoI, Theorem 4.1.9)): Let  $\{v_j\}$  be a sequence of subharmonic functions in an open set  $X \subset \mathbb{R}^m$  which have a uniform upper bound on any compact set. Then either  $v_j \rightarrow -\infty$  uniformly on every compact set, or else there exists a subsequence  $v_{j_k}$  which is convergent to some  $u \in L^1_{loc}(X)$ . Further,  $\limsup_n u_n(x) \leq u(x)$  with equality almost everywhere. For every compact subset  $K \subset X$  and every continuous function  $f$ ,

$$\limsup_{n \rightarrow \infty} \sup_K (u_n - f) \leq \sup_K (u - f).$$

In particular, if  $f \geq u$  and  $\varepsilon > 0$ , then  $u_n \leq f + \varepsilon$  on  $K$  for  $n$  large enough.

**LOGWEAK\***

**4. A general weak\* limit problem for logarithms of Husimi functions.** The study of exponential growth rates gives rise to a new kind new weak\* limit problem for complexified eigenfunctions.

**Problem 8.2.** Find the weak\* (in fact,  $L^1$ ) limits  $G$  on  $M_\varepsilon$  of sequences

$$\frac{1}{\lambda_{j_k}} \log |\varphi_{j_k}^{\mathbb{C}}(z)|^2 \rightarrow G.$$

~~ALSO LOG LIMIT GROWWEAK~~

See Theorems ~~??, ?? and ??~~ for the solution to this problem (modulo sparse subsequences) in the ergodic case.

Here is a general Heuristic principle to pin down the possible  $G$ : If  $\frac{1}{\lambda_{j_k}} \log |\varphi_{j_k}^{\mathbb{C}}(z)|^2 \rightarrow G(z)$  then

$$|\varphi_{j_k}^{\mathbb{C}}(z)|^2 \simeq e^{\lambda_{j_k} G(z)} (1 + \text{SOMETHING SMALLER}) \quad (\lambda_j \rightarrow \infty).$$

But  $\Delta_{\mathbb{C}} |\varphi_{j_k}^{\mathbb{C}}(z)|^2 = \lambda_{j_k}^2 |\varphi_{j_k}^{\mathbb{C}}(z)|^2$ , so we should have

**Conjecture 8.3.** Any limit  $G$  as above solves the Hamilton-Jacobi equation,

$$(\nabla_{\mathbb{C}} G)^2 = 1.$$

(Note: The weak\* limits of  $\frac{|\varphi_j^{\mathbb{C}}(z)|^2}{\|\varphi_j^{\mathbb{C}}\|_{L^2(\partial M_\varepsilon)}} d\mu_\varepsilon$  must be supported in  $\{G = G_{\max}\}$  (i.e. in the set of maximum values).

**5. Real zeros and complex analysis.** A natural but rather intractable problem to obtain the distribution of real zeros from knowledge of the complex nodal distribution. There exist few if any general results on this problem. In the next section we explain how to get upper bounds on real zeros using complex zeros.

It is possible to obtain results on complex zeros which are within  $\lambda^{-1}$  of the real domain by re-scaling the nodal set by a factor of  $\lambda^{-1}$  in  $M_\tau$ . But we cannot distinguish such ‘almost real zeros’ from real zeros.

It would be interesting to understand (at least in real dimension 2) how the complex nodal set ‘sprouts’ from the real nodal set. How do the connected components of the real nodal set fit together in the complex nodal set?

DFUBSECT

## 9. PROOF OF THE DONNELLY-FEFFERMAN UPPER BOUND

To prove Theorem [I.1](#), we use Crofton’s formula and a multi-dimensional Jensen’s formula to give an upper bound for  $\mathcal{H}^{n-2}(\mathcal{N}_\lambda)$  in terms of the integral geometry of  $\mathcal{N}_\lambda^{\mathbb{C}}$ . The integral geometric approach to the upper bound is inspired by the classic paper of Donnelly-Fefferman [\[DG\]](#) (see also [\[Lin\]](#)). But, instead of doubling estimates or frequency function estimates, we use the Poisson wave kernel to obtain growth estimates on eigenfunctions, and then use results on pluri-subharmonic functions rather than functions of one complex variable to relate growth of zeros to growth of eigenfunctions. This approach was used in [\[Ze07\]](#) to prove equidistribution theorems for complex nodal sets when the geodesic flow is ergodic. The Poisson wave kernel approach works for Steklov eigenfunctions as well as Laplace eigenfunctions, and in fact for eigenfunctions of any positive elliptic analytic pseudo-differential operator.

We first use the Poisson wave group [\(4.15\)](#) to analytically continue eigenfunctions in the form [\(4.3\)](#),

$$(9.1) \quad U_{\mathbb{C}}(i\tau)\psi_j(\zeta) = e^{-\tau\lambda_j}\psi_j^{\mathbb{C}}(\zeta).$$

We then use [\(9.1\)](#) to determine the growth properties of  $\psi_j^{\mathbb{C}}(\zeta)$  in Grauert tubes of the complexification of  $\partial\Omega$ . The relevant notion of Grauert tube is the standard Grauert tube for  $\partial\Omega$  with the metric  $g_{\partial\Omega}$  induced by the ambient metric  $g$  on  $M$ . This is because the principal symbol of  $\Lambda$  is the same as the principal symbol of  $\sqrt{\Delta_{\partial\Omega}}$ .

**1. Proof of Theorem [I.1](#).** We start with the integral geometric approach of [\[DF\]](#) (Lemma 6.3) (see also [\[Lin\]](#) (3.21)). There exists a ‘Crofton formula’ in the real domain which bounds the local nodal hypersurface volume above,

INTGEOM

$$(9.2) \quad \mathcal{H}^{m-1}(\mathcal{N}_{\varphi_\lambda} \cap U) \leq C_L \int_{\mathcal{L}} \#\{\mathcal{N}_{\varphi_\lambda} \cap \ell\} d\mu(\ell).$$

Thus,  $\mathcal{H}^{m-1}(\mathcal{N}_{\varphi_\lambda} \cap U)$  is bounded above by a constant  $C_L$  times the average over all line segments of length  $L$  in a local coordinate patch  $U$  of the number of intersection points of the line with the nodal hypersurface. The measure  $d\mu_L$  is known as the ‘kinematic measure’ in the Euclidean setting [\[F\]](#) (Chapter 3); see also Theorem 5.5 of [\[AP\]](#). We will be using geodesic segments of fixed length  $L$  rather than line segments, and parametrize them by

$S^*M \times [0, L]$ , i.e. by their initial data and time. Then  $d\mu_\ell$  is essentially Liouville measure  $d\mu_L$  on  $S^*M$  times  $dt$ .

The complexification of a real line  $\ell = x + \mathbb{R}v$  with  $x, v \in \mathbb{R}^m$  is  $\ell_{\mathbb{C}} = x + \mathbb{C}v$ . Since the number of intersection points (or zeros) only increases if we count complex intersections, we have

$$\boxed{\text{INEQ1}} \quad (9.3) \quad \int_{\mathcal{L}} \#(\mathcal{N}_{\varphi_\lambda} \cap \ell) d\mu(\ell) \leq \int_{\mathcal{L}} \#(\mathcal{N}_{\varphi_\lambda}^{\mathbb{C}} \cap \ell_{\mathbb{C}}) d\mu(\ell).$$

Note that this complexification is quite different from using intersections with all complex lines to measure complex nodal volumes. If we did that, we would obtain a similar upper bound on the complex hypersurface volume of the complex nodal set. But it would not give an upper bound on the real nodal volume and indeed would the complex volume tends to zero as one shrinks the Grauert tube radius to zero, while  $(9.3)$  stays bounded below.

Hence to prove Theorem 1.1 it suffices to show

$\boxed{\text{DF2}}$  **Lemma 9.1.** *We have,*

$$\mathcal{H}^{m-1}(\mathcal{N}_{\varphi_\lambda}) \leq C_L \int_{\mathcal{L}} \#(\mathcal{N}_{\varphi_\lambda}^{\mathbb{C}} \cap \ell_{\mathbb{C}}) d\mu(\ell) \leq C\lambda.$$

We now sketch the proofs of these results using a somewhat novel approach to the integral geometry and complex analysis.

$\boxed{\text{GEOS}}$  **2. Background on hypersurfaces and geodesics.** The proof of the Crofton formula given below in Lemma 9.5 involves the geometry of geodesics and hypersurfaces. To prepare for it we provide the relevant background.

As above, we denote by  $d\mu_L$  the Liouville measure on  $S^*M$ . We also denote by  $\omega$  the standard symplectic form on  $T^*M$  and by  $\alpha$  the canonical one form. Then  $d\mu_L = \omega^{n-1} \wedge \alpha$  on  $S^*M$ . Indeed,  $d\mu_L$  is characterized by the formula  $d\mu_L \wedge dH = \omega^m$ , where  $H(x, \xi) = |\xi|_g$ . So it suffices to verify that  $\alpha \wedge dH = \omega$  on  $S^*M$ . We take the interior product  $\iota_{\Xi_H}$  with the Hamilton vector field  $\Xi_H$  on both sides, and the identity follows from the fact that  $\alpha(\Xi_H) = \sum_j \xi_j \frac{\partial H}{\partial \xi_j} = H = 1$  on  $S^*M$ , since  $H$  is homogeneous of degree one. Henceforth we denote by  $\Xi = \Xi_H$  the generator of the geodesic flow.

Let  $N \subset M$  be a smooth hypersurface in a Riemannian manifold  $(M, g)$ . We denote by  $T_N^*M$  the of covectors with footpoint on  $N$  and  $S_N^*M$  the unit covectors along  $N$ . We introduce Fermi normal coordinates  $(s, y_n)$  along  $N$ , where  $s$  are coordinates on  $N$  and  $y_n$  is the normal coordinate, so that  $y_n = 0$  is a local defining function for  $N$ . We also let  $\sigma, \xi_m$  be the dual symplectic Darboux coordinates. Thus the canonical symplectic form is  $\omega_{T^*M} = ds \wedge d\sigma + dy_n \wedge d\xi_m$ . Let  $\pi : T^*M \rightarrow M$  be the natural projection. For notational simplicity we denote  $\pi^*y_n$  by  $y_n$  as functions on  $T^*M$ . Then  $y_n$  is a defining function of  $T_N^*M$ .

The hypersurface  $S_N^*M \subset S^*M$  is a kind of Poincaré section or symplectic transversal to the orbits of  $G^t$ , i.e. is a symplectic transversal away from the (at most codimension one) set of  $(y, \eta) \in S_N^*M$  for which  $\Xi_{y, \eta} \in T_{y, \eta} S_N^*M$ , where as above  $\Xi$  is the generator of the geodesic flow. More precisely,

$\boxed{\text{NSYMP}}$  **Lemma 9.2.** *The restriction  $\omega|_{S_N^*M}$  is symplectic on  $S_N^*M \setminus S^*N$ .*

Indeed,  $\omega|_{S_N^*M}$  is symplectic on  $T_{y,\eta}S^*N$  as long as  $T_{y,\eta}S_N^*M$  is transverse to  $\Xi_{y,\eta}$ , since  $\ker(\omega|_{S^*M}) = \mathbb{R}\Xi$ . But  $S^*N$  is the set of points of  $S_N^*M$  where  $\Xi \in TS_N^*M$ , i.e. where  $S_N^*M$  fails to be transverse to  $G^t$ . Indeed, transversality fails when  $\Xi(y_m) = dy_m(\Xi) = 0$ , and  $\ker dy_m \cap \ker dH = TS_N^*M$ . One may also see it in Riemannian terms as follows: the generator  $\Xi_{y,\eta}$  is the horizontal lift  $\eta^h$  of  $\eta$  to  $(y,\eta)$  with respect to the Riemannian connection on  $S^*M$ , where we freely identify covectors and vectors by the metric. Lack of transversality occurs when  $\eta^h$  is tangent to  $T_{(y,\eta)}(S_N^*M)$ . The latter is the kernel of  $dy_m$ . But  $dy_m(\eta^h) = dy_m(\eta) = 0$  if and only if  $\eta \in TN$ .

It follows from Lemma <sup>MSYMP</sup>9.2 that the symplectic volume form of  $S_N^*M \setminus S^*N$  is  $\omega^{n-1}|_{S_N^*M}$ . The following Lemma gives a useful alternative formula:

dmuLN **Lemma 9.3.** *Define*

$$d\mu_{L,N} = \iota_{\Xi} d\mu_L |_{S_N^*M},$$

where as above,  $d\mu_L$  is Liouville measure on  $S^*M$ . Then

$$d\mu_{L,N} = \omega^{m-1}|_{S_N^*M}.$$

Indeed,  $d\mu_L = \omega^{m-1} \wedge \alpha$ , and  $\iota_{\Xi} d\mu_L = \omega^{m-1}$ .

COR **Corollary 9.4.**  $\mathcal{H}^{m-1}(N) = \frac{1}{\beta_m} \int_{S_N^*M} |\omega^{m-1}|.$

**3. Hausdorff measure and Crofton formula for real geodesic arcs.** First we sketch a proof of the integral geometry estimate using geodesic arcs rather than local coordinate line segments. For background on integral geometry and Crofton type formulae we refer to <sup>AP, AP2</sup>[AP, AP2]. As explained there, a Crofton formula arises from a double fibration

$$\begin{array}{ccc} & \mathcal{I} & \\ & \swarrow \quad \searrow & \\ \pi_1 & & \pi_2 \\ & \Gamma & B, \end{array}$$

where  $\Gamma$  parametrizes a family of submanifolds  $B_\gamma$  of  $B$ . The points  $b \in B$  then parametrize a family of submanifolds  $\Gamma_b = \{\gamma \in \Gamma : b \in B_\gamma\}$  and the top space is the incidence relation in  $B \times \Gamma$  that  $b \in B_\gamma$ .

We would like to define  $\Gamma$  as the space of geodesics of  $(M, g)$ , i.e. the space of orbits of the geodesic flow on  $S^*M$ . Heuristically, the space of geodesics is the quotient space  $S^*M/\mathbb{R}$  where  $\mathbb{R}$  acts by the geodesic flow  $G^t$  (i.e. the Hamiltonian flow of  $H$ ). Of course, for a general (i.e. non-Zoll)  $(M, g)$  the ‘space of geodesics’ is not a Hausdorff space and so we do not have a simple analogue of the space of lines in  $\mathbb{R}^n$ . Instead we consider the space  $\mathcal{G}_T$  of geodesic arcs of length  $T$ . If we only use partial orbits of length  $T$ , no two partial orbits are equivalent and the space of geodesic arcs  $\gamma_{x,\xi}^T$  of length  $T$  is simply parametrized by  $S^*M$ . Hence we let  $B = S^*M$  and also  $\mathcal{G}_T \simeq S^*M$ . The fact that different arcs of length  $T$  of the same geodesic are distinguished leads to some redundancy.

In the following, let  $L_1$  denote the length of the shortest closed geodesic of  $(M, g)$ .

CROFTONEST

**Proposition 9.5.** *Let  $N \subset M$  be any smooth hypersurface<sup>1</sup>, and let  $S_N^*M$  denote the unit covers to  $M$  with footpoint on  $N$ . Then for  $0 < T < L_1$ ,*

$$\mathcal{H}^{m-1}(N) = \frac{1}{\beta_m T} \int_{S^*M} \#\{t \in [-T, T] : G^t(x, \omega) \in S_N^*M\} d\mu_L(x, \omega),$$

where  $\beta_m$  is  $2(m-1)!$  times the volume of the unit ball in  $\mathbb{R}^{m-2}$ .

*Proof.* By Corollary <sup>COR</sup>9.4, the Hausdorff measure of  $N$  is given by

HNN

$$(9.4) \quad \mathcal{H}^{m-1}(N) = \frac{1}{\beta_m} \int_{S_N^*M} |\omega^{m-1}|.$$

We use the Lagrange (or more accurately, Legendre) immersion,

$$\iota : S^*M \times \mathbb{R} \rightarrow S^*M \times S^*M, \quad \iota(x, \omega, t) = (x, \omega, G^t(x, \omega)),$$

where as above,  $G^t$  is the geodesic flow. We also let  $\pi : T^*M \rightarrow M$  be the standard projection. We restrict  $\iota$  to  $S^*M \times [-T, T]$  and define the incidence relation

$$\mathcal{I}_T = \{((y, \eta), (x, \omega), t) \subset S^*M \times S^*M \times [-T, T] : (y, \eta) = G^t(x, \omega)\},$$

which is isomorphic to  $[-T, T] \times S^*M$  under  $\iota$ . We form the diagram

$$\begin{array}{ccc} & \pi_1 \swarrow & \searrow \pi_2 \\ S^*M \simeq \mathcal{G}_T & & S^*M, \end{array}$$

using the two natural projections, which in the local parametrization take the form

$$\pi_1(t, x, \xi) = G^t(x, \xi), \quad \pi_2(t, x, \xi) = (x, \xi).$$

As noted above, the bottom left  $S^*M$  should be thought of as the space of geodesic arcs. The fiber

$$\pi_1^{-1}(y, \eta) = \{(t, x, \xi) \in [-T, T] \times S^*M : G^t(x, \xi) = (y, \eta)\} \simeq \gamma_{(y, \eta)}^T$$

may be identified with the geodesic segment through  $(y, \eta)$  and the fiber  $\pi_2^{-1}(x, \omega) \simeq [-T, T]$ .

We ‘restrict’ the diagram above to  $S_N^*M$ :

DIAGRAM

$$(9.5) \quad \begin{array}{ccc} & \pi_1 \swarrow & \searrow \pi_2 \\ (S_N^*M)_T & & S_N^*M, \end{array}$$

where

$$(S_N^*M)_T = \pi_1 \pi_2^{-1}(S_N^*M) = \bigcup_{|t| < T} G^t(S_N^*M).$$

We define the Crofton density  $\varphi_T$  on  $S_N^*M$  corresponding to the diagram <sup>DIAGRAM</sup>(9.5) [AP] (section 4) by

CROFDEN

$$(9.6) \quad \varphi_T = (\pi_2)_* \pi_1^* d\mu_L.$$

<sup>1</sup>The same formula is true if  $N$  has a singular set  $\Sigma$  with  $\mathcal{H}^{m-2}(\Sigma) < \infty$

Since the fibers of  $\pi_2$  are 1-dimensional,  $\varphi_T$  is a differential form of dimension  $2 \dim M - 2$  on  $S^*M$ . To make it smoother, we can introduce a smooth cutoff  $\chi$  to  $(-1, 1)$ , equal to 1 on  $(-\frac{1}{2}, \frac{1}{2})$ , and use  $\chi_T(t) = \chi(\frac{t}{T})$ . Then  $\pi_1^*(d\mu_L \otimes \chi_T dt)$  is a smooth density on  $\mathcal{I}_T$ .

**phiT** **Lemma 9.6.** *The Crofton density (9.6) is given by,  $\varphi_T = T d\mu_{L,N}$*

*Proof.* In (9.5) we defined the map  $\pi_1 : (y, \eta, t) \in S_N^*M \times [-T, T] \rightarrow G^t(y, \eta) \in (S^*M)_\varepsilon$ . We first claim that  $\pi_1^* d\mu_L = d\mu_{L,N} \otimes dt$ . This is essentially the same as Lemma 9.3. Indeed,  $d\pi_1(\frac{\partial}{\partial t}) = \Xi$ , hence  $\iota_{\frac{\partial}{\partial t}} \pi_1^* d\mu_L|_{(t,y,\eta)} = (G^t)^* \omega^{m-1} = \omega^{m-1}|_{T_{y,\eta} S_N^*M}$ .

Combining Lemma 9.6 with (9.4) gives

**HDPHIT** (9.7) 
$$\int_{S_N^*M} \varphi_T = \int_{\pi_2^{-1}(S_N^*M)} d\mu_L = T \beta_m \mathcal{H}^{m-1}(N).$$

□

We then relate the integral on the left side to numbers of intersections of geodesic arcs with  $N$ . The relation is given by the co-area formula: if  $f : X \rightarrow Y$  is a smooth map of manifolds of the same dimension and if  $\Phi$  is a smooth density on  $Y$ , and if  $\#\{f^{-1}(y)\} < \infty$  for every regular value  $y$ , then

$$\int_X f^* \Phi = \int_Y \#\{f^{-1}(y)\} \Phi.$$

If we set  $X = \pi_2^{-1}(S_N^*M)$ ,  $Y = S^*M$ , and  $f = \pi_1|_{\pi_2^{-1}(S_N^*M)}$  then the co-area formula gives,

**COAREA** (9.8) 
$$\int_{\pi_2^{-1}(S_N^*M)} \pi_1^* d\mu_L = \int_{S^*M} \#\{t \in [-T, T] : G^t(x, \omega) \in S_N^*M\} d\mu_L(x, \omega).$$

Combining (9.7) and (9.8) gives the result stated in Proposition 9.5,

**CONCLUSION** (9.9) 
$$T \beta_m \mathcal{H}^{m-1}(N) = \int_{S^*M} \#\{t \in [-T, T] : G^t(x, \omega) \in S_N^*M\} d\mu_L(x, \omega).$$

□

**4. Proof of Lemma 9.1.** <sup>DF2</sup> The next step is to complexify.

*Proof.* We complexify the Lagrange immersion  $\iota$  from a line (segment) to a strip in  $\mathbb{C}$ : Define

$$F : S_\varepsilon \times S^*M \rightarrow M_{\mathbb{C}}, \quad F(t + i\tau, x, v) = \exp_x(t + i\tau)v, \quad (|\tau| \leq \varepsilon)$$

By definition of the Grauert tube,  $\psi$  is surjective onto  $M_\varepsilon$ . For each  $(x, v) \in S^*M$ ,

$$F_{x,v}(t + i\tau) = \exp_x(t + i\tau)v$$

is a holomorphic strip. Here,  $S_\varepsilon = \{t + i\tau \in \mathbb{C} : |\tau| \leq \varepsilon\}$ . We also denote by  $S_{\varepsilon,L} = \{t + i\tau \in \mathbb{C} : |\tau| \leq \varepsilon, |t| \leq L\}$ .

Since  $F_{x,v}$  is a holomorphic strip,

$$F_{x,v}^* \left( \frac{1}{\lambda} dd^c \log |\psi_j^{\mathbb{C}}|^2 \right) = \frac{1}{\lambda} dd_{t+i\tau}^c \log |\psi_j^{\mathbb{C}}|^2(\exp_x(t + i\tau)v) = \frac{1}{\lambda} \sum_{t+i\tau: \psi_j^{\mathbb{C}}(\exp_x(t+i\tau)v)=0} \delta_{t+i\tau}.$$

Put:

$$\boxed{\text{acal}} \quad (9.10) \quad \mathcal{A}_{L,\varepsilon} \left( \frac{1}{\lambda} dd^c \log |\psi_j^{\mathbb{C}}|^2 \right) = \frac{1}{\lambda} \int_{S^*M} \int_{S_{\varepsilon,L}} dd_{t+i\tau}^c \log |\psi_j^{\mathbb{C}}|^2 (\exp_x(t+i\tau)v) d\mu_L(x,v).$$

A key observation of  $\boxed{\text{DF, Lin}}$  is that

$$\boxed{\text{MORE}} \quad (9.11) \quad \#\{\mathcal{N}_\lambda^{\mathbb{C}} \cap F_{x,v}(S_{\varepsilon,L})\} \geq \#\{\mathcal{N}_\lambda^{\mathbb{R}} \cap F_{x,v}(S_{0,L})\},$$

since every real zero is a complex zero. It follows then from Proposition  $\boxed{\text{CROFTONEST}}$  9.5 (with  $N = \mathcal{N}_\lambda$ ) that

$$\begin{aligned} \mathcal{A}_{L,\varepsilon} \left( \frac{1}{\lambda} dd^c \log |\psi_j^{\mathbb{C}}|^2 \right) &= \frac{1}{\lambda} \int_{S^*M} \#\{\mathcal{N}_\lambda^{\mathbb{C}} \cap F_{x,v}(S_{\varepsilon,L})\} d\mu(x,v) \\ &\geq \frac{1}{\lambda} \mathcal{H}^{m-1}(\mathcal{N}_{\psi_\lambda}). \end{aligned}$$

Hence to obtain an upper bound on  $\frac{1}{\lambda} \mathcal{H}^{m-1}(\mathcal{N}_{\psi_\lambda})$  it suffices to prove that there exists  $M < \infty$  so that

$$\boxed{\text{acalest}} \quad (9.12) \quad \mathcal{A}_{L,\varepsilon} \left( \frac{1}{\lambda} dd^c \log |\psi_j^{\mathbb{C}}|^2 \right) \leq M.$$

To prove  $\boxed{\text{acalest}}$  (9.12), we observe that since  $dd_{t+i\tau}^c \log |\psi_j^{\mathbb{C}}|^2 (\exp_x(t+i\tau)v)$  is a positive  $(1,1)$  form on the strip, the integral over  $S_\varepsilon$  is only increased if we integrate against a positive smooth test function  $\chi_\varepsilon \in C_c^\infty(\mathbb{C})$  which equals one on  $S_{\varepsilon,L}$  and vanishes off  $S_{2\varepsilon,L}$ . Integrating by parts the  $dd^c$  onto  $\chi_\varepsilon$ , we have

$$\begin{aligned} \mathcal{A}_{L,\varepsilon} \left( \frac{1}{\lambda} dd^c \log |\psi_j^{\mathbb{C}}|^2 \right) &\leq \frac{1}{\lambda} \int_{S^*M} \int_{\mathbb{C}} dd_{t+i\tau}^c \log |\psi_j^{\mathbb{C}}|^2 (\exp_x(t+i\tau)v) \chi_\varepsilon(t+i\tau) d\mu_L(x,v) \\ &= \frac{1}{\lambda} \int_{S^*M} \int_{\mathbb{C}} \log |\psi_j^{\mathbb{C}}|^2 (\exp_x(t+i\tau)v) dd_{t+i\tau}^c \chi_\varepsilon(t+i\tau) d\mu_L(x,v). \end{aligned}$$

Now write  $\log |x| = \log_+ |x| - \log_- |x|$ . Here  $\log_+ |x| = \max\{0, \log |x|\}$  and  $\log_- |x| = \max\{0, -\log |x|\}$ . Then we need upper bounds for

$$\frac{1}{\lambda} \int_{S^*M} \int_{\mathbb{C}} \log_\pm |\psi_j^{\mathbb{C}}|^2 (\exp_x(t+i\tau)v) dd_{t+i\tau}^c \chi_\varepsilon(t+i\tau) d\mu_L(x,v).$$

For  $\log_+$  the upper bound is an immediate consequence of Proposition  $\boxed{\text{PW}}$  4.3. For  $\log_-$  the bound is subtler: we need to show that  $|\varphi_\lambda(z)|$  cannot be too small on too large a set. As we know from Gaussian beams, it is possible that  $|\varphi_\lambda(x)| \leq Ce^{-\delta\lambda}$  on sets of almost full measure in the real domain; we need to show that nothing worse can happen.

The map  $\boxed{\text{EXP}}$  (2.3) is a diffeomorphism and since  $B_\varepsilon^*M = \bigcup_{0 \leq \tau \leq \varepsilon} S_\tau^*M$  we also have that

$$E : S_{\varepsilon,L} \times S^*M \rightarrow M_\tau, \quad E(t+i\tau, x, v) = \exp_x(t+i\tau)v$$

is a diffeomorphism for each fixed  $t$ . Hence by letting  $t$  vary,  $E$  is a smooth fibration with fibers given by geodesic arcs. Over a point  $\zeta \in M_\tau$  the fiber of the map is a geodesic arc

$$\{(t+i\tau, x, v) : \exp_x(t+i\tau)v = \zeta, \tau = \sqrt{\rho}(\zeta)\}.$$

Pushing forward the measure  $dd_{t+i\tau}^c \chi_\varepsilon(t+i\tau) d\mu_L(x,v)$  under  $E$  gives a positive measure  $d\mu$  on  $M_\tau$ . We claim that

$$\boxed{\text{PUSH}} \quad (9.13) \quad \mu := E_* dd_{t+i\tau}^c \chi_\varepsilon(t+i\tau) d\mu_L(x,v) = \left( \int_{\gamma_{x,v}} \Delta_{t+i\tau} \chi_\varepsilon ds \right) dV_\omega,$$

where  $dV_\omega$  is the Kähler volume form  $\frac{\omega^m}{m!}$ .

In fact,  $d\mu_L$  is equivalent under  $E$  to the contact volume form  $\alpha \wedge \omega_\rho^{m-1}$  where  $\alpha = d^c \sqrt{\rho}$ . Hence the claim amounts to saying that the Kähler volume form is  $d\tau$  times the contact volume form. In particular it is a smooth (and of course signed) multiple  $J$  of the Kähler volume form  $dV_\omega$ , and we do not need to know the coefficient function  $J$  beyond that it is bounded above and below by constants independent of  $\lambda$ . We then have

$$\boxed{\text{JEN}} \quad (9.14) \quad \int_{S^*M} \int_{\mathbb{C}} \log |\psi_j^{\mathbb{C}}|^2 (\exp_x(t + i\tau)v) dd_{t+i\tau}^c \chi_\varepsilon(t + i\tau) d\mu_L(x, v) = \int_{M_\tau} \log |\psi_j^{\mathbb{C}}|^2 J dV.$$

To complete the proof of <sup>acalest</sup>(9.12) it suffices to prove that the right side is  $\geq -C\lambda$  for some  $C > 0$ .

We use the well-known Lemma <sup>HARTOGS</sup>8.1. This Lemma implies the desired lower bound on <sup>JEN</sup>(9.14): there exists  $C > 0$  so that

$$\boxed{\text{LOGINT}} \quad (9.15) \quad \frac{1}{\lambda} \int_{M_\tau} \log |\psi_\lambda| J dV \geq -C.$$

For if not, there exists a subsequence of eigenvalues  $\lambda_{j_k}$  so that  $\frac{1}{\lambda_{j_k}} \int_{M_\tau} \log |\psi_{\lambda_{j_k}}| J dV \rightarrow -\infty$ . By Proposition <sup>PW</sup>4.3,  $\{\frac{1}{\lambda_{j_k}} \log |\psi_{\lambda_{j_k}}|\}$  has a uniform upper bound. Moreover the sequence does not tend uniformly to  $-\infty$  since  $\|\psi_\lambda\|_{L^2(M)} = 1$ . It follows that a further subsequence tends in  $L^1$  to a limit  $u$  and by the dominated convergence theorem the limit of <sup>LOGINT</sup>(9.15) along the sequence equals  $\int_{M_\tau} u J dV \neq -\infty$ . This contradiction concludes the proof of <sup>LOGINT</sup>(9.15), hence <sup>acalest</sup>(9.12), and thus the theorem. □

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